Special Relativity

Particle Dynamics

Required for module PHY304 Particle Physics and possible the problem solving paper of PHY340

Experiments reveal that the mass, m, of an object increases with speed, v, and is given by:

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 \quad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is the mass in a reference frame where the object is not moving, $v=0$.

The energy and momentum of the body are given by

$$E = mc^2 = \gamma m_0 c^2$$

$$\mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$$

When solving dynamical problems we still use conservation of energy and momentum but if the body is travelling at speeds comparable to *c* we have to use the above equations for *E* and **p**. This is normally needed for problems involving sub-atomic particles

Remember that **p** is a vector quantity.

From the two equations

$$E = mc^2 = \gamma m_0 c^2 \qquad \mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$$

We can derive the following relationship that eliminates *v*

$$E^2 = c^2 p^2 + (m_0 c^2)^2$$

Here p^2 is the magnitude squared of the momentum vector and *E* is the total energy

Because the equation

$$E^2 = c^2 p^2 + (m_0 c^2)^2$$

doesn't contain *v* it is very useful for many problems.

Rearranging we get:

$$E^2 - c^2 p^2 = (m_0 c^2)^2$$

As the right hand side is a constant the left hand side has the same value (it is an invariant) in different frames of reference.

E and *p* can refer to the energy and momentum of a single particle or the sums for all the particles in a particular frame

What is the relationship between energy and momentum (E and p) for a photon?

We use $E^2 = c^2 p^2 + (m_0 c^2)^2$ and the fact that a photon is massless m_0=0

So $E^2 = c^2 p^2 \Rightarrow E = cp$

Other useful relationships

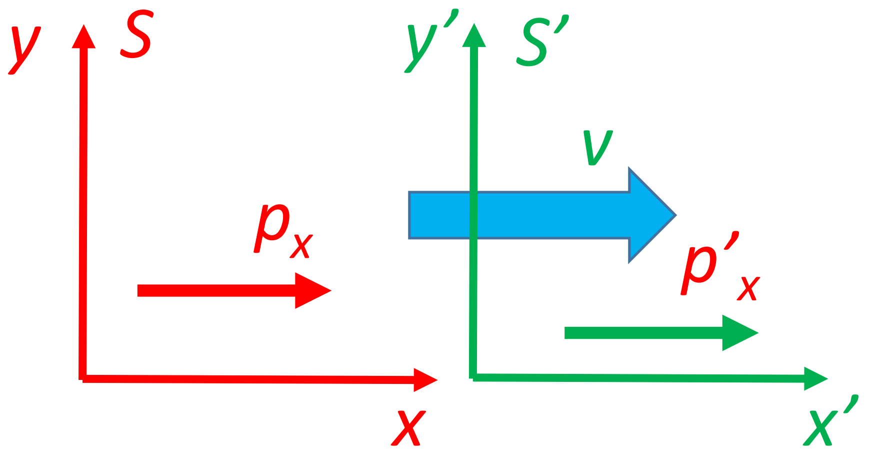
If we have two frames of reference *S* and *S'* with *S'* moving with a velocity *v* with respect to *S* along the *x*-axis then we can related the energy and momentum in *S* (*E* and *p*) to the values in *S'*(*E'* and *p'*) (and the other way – vice versa)

$$p_x' = \gamma(p_x - vE/c^2) \qquad p_x = \gamma(p_x' + vE'/c^2)$$

$$p_y' = p_y \qquad p_y = p_y'$$

$$p_z' = p_z \qquad p_z = p_z'$$

$$E' = \gamma(E - vp_x) \qquad E = \gamma(E' + vp_x')$$



The only other equation that might be needed is the one giving the relationship between velocities in different frames of reference

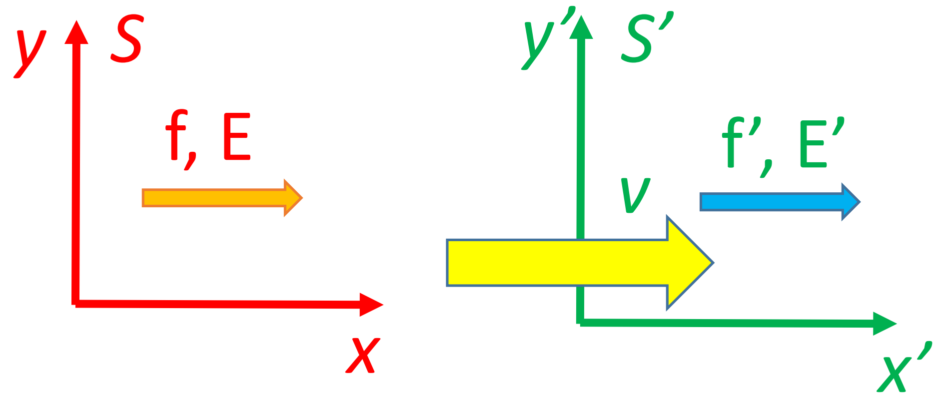
If an object has velocity *u′* in frame *S′* then

$$u = \frac{u' + v}{1 + uv/c^2} \qquad u' = \frac{u - v}{1 - uv/c^2}$$

Applications: 1) the Doppler effect

A source emits a photon of energy E, frequency f in its rest frame. What are the values of E and f observed in a frame which the source is moving towards with velocity v?

Take S as the frame in which the source is at rest and transform to the frame S'. S' has a relative velocity of v with respect to S. Assume the photon is emitted in the positive x direction.



In S: $p_x = E/c$ for a photon

Using $E' = \gamma(E - vp_x) = \gamma(E - (v/c)E) = \gamma E(1 - v/c)$

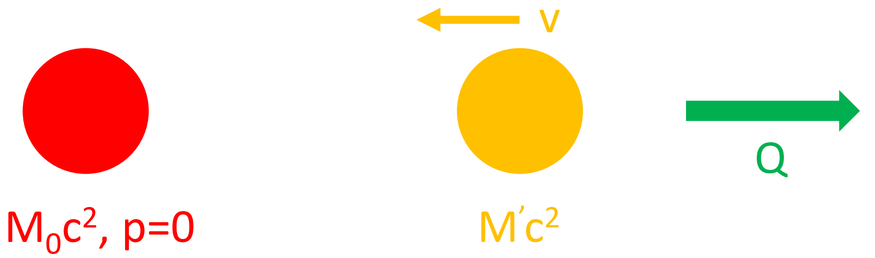
$$E' = \gamma E(1 - v/c) = E\frac{(1 - v/c)}{\sqrt{1 - (v/c)^2}} = E\frac{(1 - v/c)}{\sqrt{1 - (v/c)}\sqrt{1 + (v/c)}} = E\sqrt{\frac{1 - v/c}{1 + v/c}}$$

We can use $E=hf$ to relate the frequencies

If the source moves away from the observer then v changes sign and the result becomes

$$E' = E\sqrt{\frac{1 + v/c}{1 - v/c}}$$

2) Emission of photons and recoil



An atom at rest of mass M_0 emits a photon of energy Q (could be a gamma ray). To conserve momentum the atom recoils. Let the recoiling atom have mass M' (and rest mass M_0'), energy E', momentum p' and velocity v

Conservation of energy (initial energy = final energy)

$$E = M_0c^2 = M'c^2 + Q = E' + Q$$

Conservation of momentum

$$p = 0 = M'v - Q/c = p' - Q/c$$

$$\Rightarrow \quad E' = M_0c^2 - Q \quad \text{and} \quad cp' = Q$$

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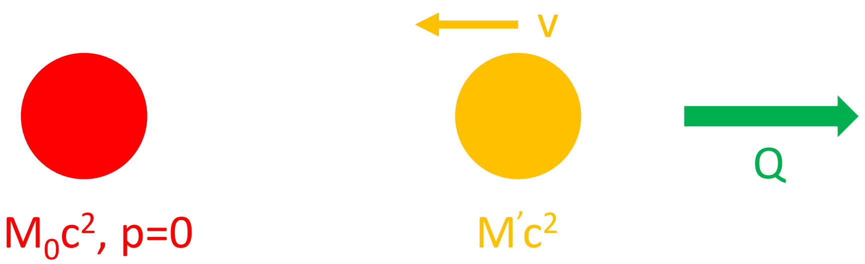


We also have

$$\left(M_0'c^2\right)^2 = (E')^2 - (cp')^2 = (M_0c^2 - Q)^2 - (Q)^2$$

$$\Rightarrow \left(M_0'c^2\right)^2 = (M_0c^2)^2 - 2M_0c^2Q \quad \text{(A)}$$

The difference in rest mass energies of the atom in the initial and final state will have a well defined quantity which we call Q_0

$$M_0c^2 - M_0'c^2 = Q_0 \quad \Rightarrow M_0'c^2 = M_0c^2 - Q_0 \quad \text{(B)}$$

Squaring (B) and equating to (A)

$$\left(M_0'c^2\right)^2=\left(M_0c^2-Q_0\right)^2=(M_0c^2)^2-2M_0c^2Q$$

$$(M_0c^2)^2-2M_0c^2Q_0+Q_0^{\ 2}=(M_0c^2)^2-2M_0c^2Q$$

$$-2M_0c^2Q_0+Q_0^{\ 2}=-2M_0c^2Q$$

$$Q_0\left(-2M_0c^2+Q_0\right)=-2M_0c^2Q$$

$$\Rightarrow\ Q=Q_0\left(1-\frac{Q_0}{2M_0c^2}\right)$$

The result

$$Q = Q_0\left(1 - \frac{Q_0}{2M_0c^2}\right)$$

shows that the emitted photon Q has less energy than is lost by the atom Q_0. This is because some of Q_0 is required to give the atom the necessary recoil velocity.

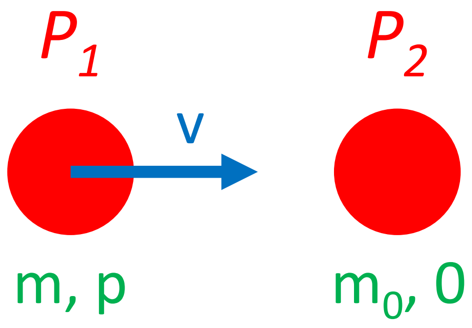
In some systems the emitted photon may not be able to be reabsorbed by the same transition as it has insufficient energy.

3 Particle production – proton-proton creation of a π meson

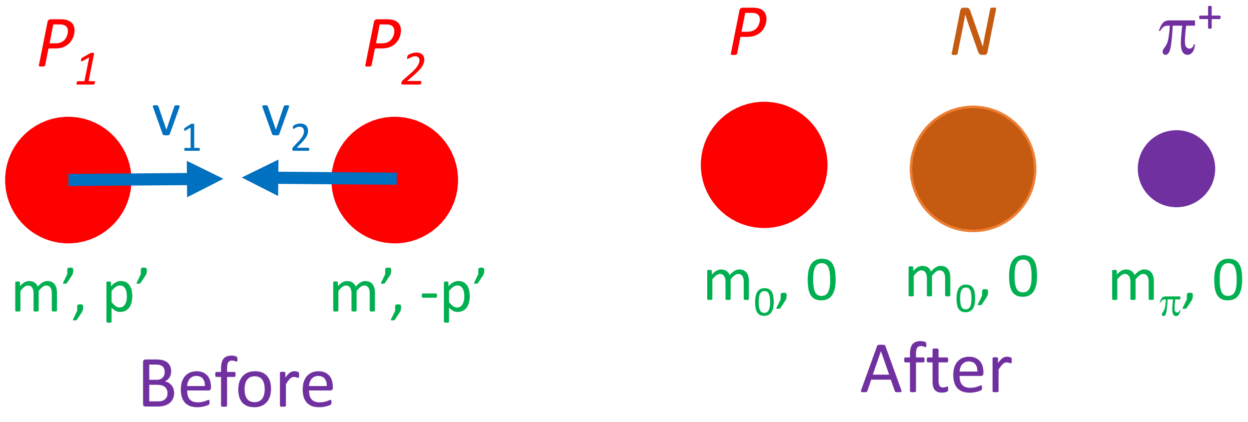
Colliding particles in an accelerator and using some of the kinetic energy to create new particles is the main technique used by particle physicists.
An important question is how much energy the particle(s) must have to create a given particle. Here we consider a typical example.

A proton P_1 is collided with a stationary proton P_2 and creates a π meson (a proton P and neutron N also remain). How much energy must P_1 have for this process to be possible. The rest mass energies of a proton and a π meson are 938 and 139.6 MeV respectively. Assume the neutron and proton masses are identical.

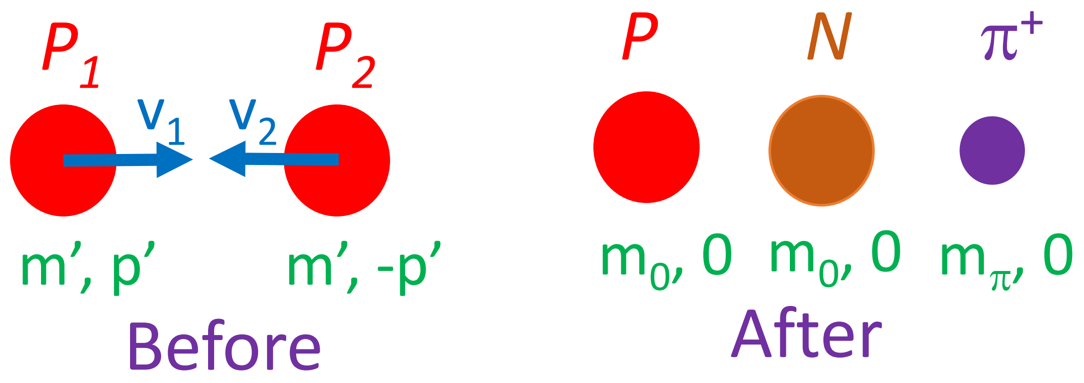
The real experiment



These problems are generally easier to solve by initially working in the centre of mass frame where the total momentum is zero. Imagine moving in a frame so that P_2 approaches P_1 with an equal but opposite velocity. This frame has the advantage that after the process we can assume that the three particles are stationary if there is just enough energy for this process to occur.



Centre of mass frame



Conservation of energy $E = 2m'c^2 = 2m_0c^2 + m_\pi c^2$

$$\Rightarrow \frac{m'}{m_0} = 1 + \frac{m_\pi}{2m_0}$$

Giving $m'/m_0 = 1.074$

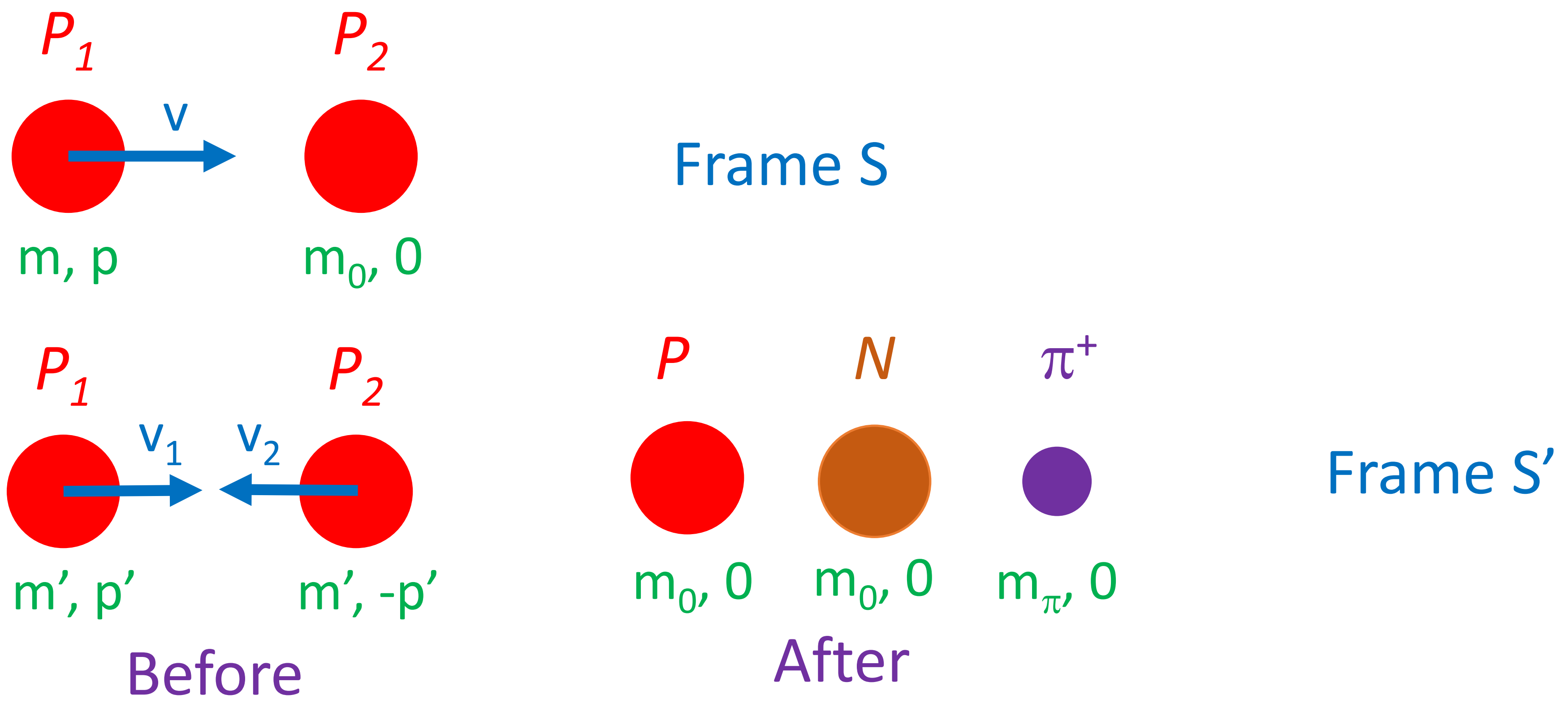
$$m'/m_0 = \gamma = (1-\beta^2)^{-1/2} \quad \Rightarrow \beta \approx 0.37$$

So in the centre of mass frame $\beta \approx 0.37$.
However in the laboratory frame P_2 is at rest so the centre of mass frame must have a velocity of β relative to the lab frame.
As P_1 has a velocity of β relative to the centre of mass frame we can find the velocity of P_1 in the lab frame using the addition of velocities formula.

$$\beta_{lab} = \frac{\beta + \beta}{1 + \beta^2} = \frac{2\beta}{1 + \beta^2} \approx 0.65 \quad \Rightarrow \gamma_{lab} = (1 - \beta_{lab}^2)^{-1/2} \approx 1.31$$

So the KE of P_1 is $0.31 m_0 c^2 \approx 290$ MeV – approximately x2 the rest mass of the π meson which is 139.6 MeV

3) Particle production – proton-proton creation of a π meson – alternative approach



We use the relationship $E^2-(cp)^2 = const = (m_0c^2)^2$ which applies both to all the total energy and momentum in a frame and also to an individual particle in a frame

In S' $E' = 2m'c^2 = 2m_0c^2 + m_\pi c^2$ $\quad p'_{tot} = 0$

In S $\quad E = (m+m_0)c^2 \quad p = p$

Evaluating and equating $E^2-(cp)^2$ for both S and S'

$$(mc^2+m_0c^2)^2-(cp)^2 = (2m_0c^2+m_\pi c^2)^2$$

$$m^2c^4+m_0^2c^4+2mm_0c^4-(cp)^2 = 4m_0^2c^4+m_\pi^2c^4+4m_0m_\pi c^4$$

$$m^2c^4 + m_0^2c^4 + 2mm_0c^4 - (cp)^2 = 4m_0^2c^4 + m_\pi^2c^4 + 4m_0m_\pi c^4$$

But for the incident proton $m^2c^4 - (cp)^2 = (m_0c^2)^2$

$$m_0^2c^4 + m_0^2c^4 + 2mm_0c^4 = 4m_0^2c^4 + m_\pi^2c^4 + 4m_0m_\pi c^4$$

$$2mm_0c^4 = 2m_0^2c^4 + m_\pi^2c^4 + 4m_0m_\pi c^4$$

$$2mm_0 = 2m_0^2 + m_\pi^2 + 4m_0m_\pi$$

$$\Rightarrow \frac{m}{m_0} = 1 + \frac{m_\pi}{m_0}\left(2 + \frac{m_\pi}{2m_0}\right) \approx 1.31$$