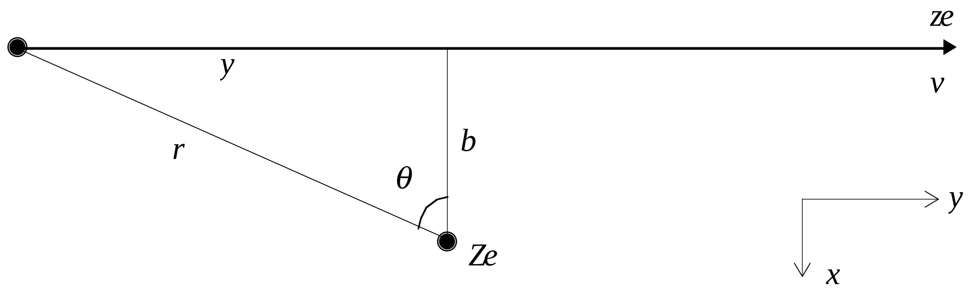
Interaction of a Charged Particle with Matter

Consider a particle, of charge ze, passing a stationary charge Ze (the "target" particle) with impact parameter b and velocity v.



Assuming that the moving particle passes the target so rapidly that the latter remains stationary during the "collision", b will remain constant. In this case, the longitudinal electrostatic force exerted on the target before and after the moment of closest approach will cancel. The effective transverse force is then

$$F_x = \frac{Zze^2}{4\pi\varepsilon_0 r^2}\cos\theta \qquad\qquad r = \frac{b}{\cos\theta}$$

$$= \frac{Zze^2}{4\pi\varepsilon_0 b^2}\cos^3\theta$$

The impulse delivered to the target is therefore

$$\mathrm{d}t = \frac{\mathrm{d}y}{v}$$

$$\Delta p = \int_{-\infty}^{\infty} F_x \mathrm{d}t \qquad\qquad y = b\tan\theta$$

$$\mathrm{d}y = b\sec^2\theta\,\mathrm{d}\theta$$

$$= \frac{Zze^2}{4\pi\varepsilon_0 b^2}\int_{-\pi/2}^{\pi/2}\cos^3\theta\,\frac{b}{v\cos^2\theta}\,\mathrm{d}\theta$$

i.e.
$$\Delta p = \frac{Zze^2}{2\pi\varepsilon_0 bv}$$

The above calculation has been performed non-relativistically. If the projectile particle is moving relativistically: E_x is increased by a factor γ

dy is decreased by a factor γ

So Δp remains unchanged as $\dfrac{Zze^2}{2\pi\varepsilon_0 b\beta c}$.

For some purposes, it is useful to consider this momentum transfer as the product of the maximum force exerted and a characteristic time.

$$\Delta p = F_x^{\max}.\tau$$

where $$F_x^{\max} = \frac{Zze^2}{4\pi\varepsilon_0 b^2}\gamma$$

and the "collision time" is therefore $$\tau = \frac{2b}{\gamma\beta c} \qquad [1]$$

Note the assumptions that have been made:

1) Impulse approximation – the target does not move (significantly) during the collision
2) The target remains non-relativistic.

If the latter is true, then the target gains kinetic energy given by

$$E_{\mathrm{T}} = \frac{(\Delta p)^2}{2m_{\mathrm{T}}} = \frac{Z^2 z^2 e^4}{2(2\pi\varepsilon_0)^2 b^2 \beta^2 c^2 m_{\mathrm{T}}} \propto \frac{Z^2}{m_{\mathrm{T}}}$$

The matter the particle travels through consists of nuclei, of charge Ze and mass (approximately) Am_p, each surrounded by Z electrons each of charge e and mass m_e.

Thus $$\frac{\text{Energy transfer to nuclei}}{\text{Energy transfer to electrons}} = \frac{Z^2/Am_p}{Z\,1/m_e} \approx \frac{Z/2m_p}{Z/m_e} = \frac{m_e}{2m_p}\ .$$

Given the ratio between the electron and proton masses, it is therefore reasonable to consider only the energy lost to electrons. For a single electron, we have

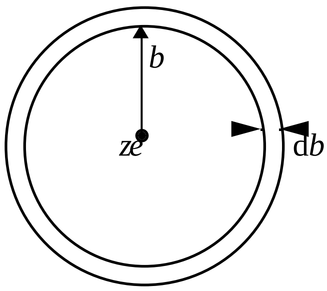
$$E_e = \frac{2z^2e^4}{(4\pi\varepsilon_0)^2 b^2 \beta^2 m_e c^2} \qquad [2]$$

Note: E_e is determined by the impact parameter b. Therefore the probability of an energy loss between E and E+dE is given by the probability of an impact parameter between b and b+db, where b corresponds to E and b+db corresponds to E+dE,

$$P(E_e)\mathrm{d}E = -P'(b)\mathrm{d}b$$

the minus sign arising as an increase in b results in a decrease in E.

The energy loss is not due to an interaction with a single target electron, however. Consider a thin slice of material, of density ρ and thickness Δx.



Then $$P'(b)\mathrm{d}b = 2\pi b\,\mathrm{d}b \times \text{no. of electrons per unit area}$$
$$= 2\pi b\ Z\frac{N_{\mathrm{A}}}{A}\rho\ \Delta x\ \mathrm{d}b$$

[2] ⇒ $$b^2 = \frac{2z^2e^4}{(4\pi\varepsilon_0)^2\beta^2 m_e c^2 E_e}$$

⇒ $$2b\ \mathrm{d}b = -\frac{2z^2e^4}{(4\pi\varepsilon_0)^2\beta^2 m_e c^2}\frac{\mathrm{d}E}{E_e{}^2}$$

⇒ $$P(E_e)\ \mathrm{d}E = \frac{\pi Z\, N_\mathrm{A}\,\rho\Delta x}{A}\frac{2z^2e^4}{(4\pi\varepsilon_0)^2\beta^2 m_e c^2}\frac{dE}{E_e{}^2} \qquad [3]$$

So the most probable energy loss in traversing this slice is

$$\overline{\Delta E} = \int_{E_\mathrm{min}}^{E_\mathrm{max}} E\ P(E)\ \mathrm{d}E \qquad [4]$$

$$= \frac{2\pi\, Z\, N_\mathrm{A}\,\rho\Delta x\, z^2 e^4}{(4\pi\varepsilon_0)^2\beta^2 m_e c^2 A}[\ln E]_{E_\mathrm{min}}^{E_\mathrm{max}}$$

This can be simplified by grouping together a number of physical constants, which depend on the properties of neither the projectile particle nor the target material. Let us define

$$C = 2\pi\, N_\mathrm{A}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2 \qquad [5]$$

i.e. $C = 0.03006\ (\mathrm{kg\ m^{-2}})^{-1} = 0.3006\ (\mathrm{g\ cm^{-2}})^{-1}$.

Hence $$\overline{\Delta E} = C\frac{m_e c^2}{\beta^2}\frac{Zz^2}{A}\rho\Delta x[\ln E]_{E_\mathrm{min}}^{E_\mathrm{max}} \qquad [6]$$

$$= 2C\frac{m_e c^2}{\beta^2}\frac{Zz^2}{A}\rho\Delta x[\ln b]_{b_\mathrm{min}}^{b_\mathrm{max}} \qquad [6']$$

Clearly there must be some limits on E_min and E_max (or b_min and b_max) to prevent the value of the integral being infinite, which is obviously unphysical! For E_min, this is where the collision becomes very "soft", and the electric field of the passing particle simply perturbs the atomic electron adiabatically, with no energy being absorbed. This occurs when the collision time is long compared with the period of the electron in its atomic orbit

i.e. $$\tau > 1/f_\mathrm{rot}$$

[1] ⇒ $$\frac{2b}{\gamma\beta c} > \frac{1}{f_\mathrm{rot}}$$

Now $h\, f_\mathrm{rot} \approx I_0$, where I_0 is the <u>mean</u> ionisation potential of the atom, so

$$b_\mathrm{max} = \frac{\gamma\beta c}{2\, f_\mathrm{rot}} \approx \frac{\gamma\beta c h}{2I_0}$$

For E_max, there are a number of possible limits. Evidently the above calculation becomes invalid when the electron receives enough energy for it to become relativistic. There are also absolute limits on the maximum transferable energy. For example, it clearly cannot be greater than the energy of the incoming particle. A proper quantum mechanical, relativistic calculation is needed for the full result. However, it can be determined approximately by relatively simple quantum mechanical arguments, and the appendix shows that for any incoming particle significantly heavier than an electron (i.e. muon, pion, proton or nuclear fragment), then the most important limit comes from the fact that the uncertainty principle does not allow b to be specified too precisely, and so limits b_min. For a projectile particle which is not <u>extremely</u> relativistic ($\gamma < 100$), then the limit on b_min is (see appendix)

$$b_{\min} \approx \frac{\hbar}{m_e \gamma \beta c}$$

Substituting these limits in to [6′] gives the following approximate expression for the mean energy loss

$$\overline{\Delta E} = 2C \frac{m_e c^2}{\beta^2} \frac{Zz^2}{A} \rho \Delta x \ln\left(\frac{\gamma \beta c h}{2 I_0} \frac{m_e \gamma \beta c}{\hbar} \right)$$

$$= 2C \frac{m_e c^2}{\beta^2} \frac{Zz^2}{A} \rho \Delta x \ln\left(\frac{\pi \gamma^2 \beta^2 m_e c}{I_0} \right)$$

The full quantum mechanical treatment (by Bethe and Bloch) for “heavy” particles (not electrons) gives the result

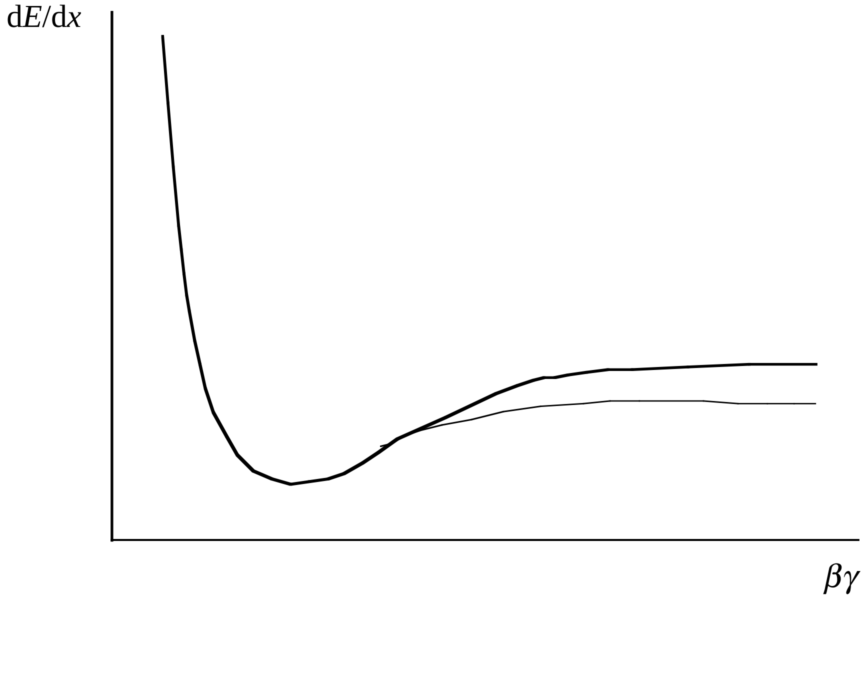
$$\boxed{\overline{\Delta E} = 2C \frac{m_e c^2}{\beta^2} \frac{Zz^2}{A} \rho \Delta x \left[\ln\left(\frac{2\gamma^2 \beta^2 m_e c}{I_0} \right) - \beta^2 - \frac{\varepsilon}{2} - \frac{\delta(\beta)}{2} \right]}$$

Note the two additional terms:

ε is a small correction due to screening of the inner atomic electrons (the “shell correction”). It is often ignored.

δ is a function of β and the dielectric constant of the medium, and is known as the “density effect”. Polarisation of the material at large values of b (which are only important for large γ) screens the effect of the traversing charge. It is much more important for dense media, such as solids, than it is for gases.

Schematically, the variation of mean energy loss per unit thickness, ${\mathrm{d}E}/{\mathrm{d}x}$, has the following behaviour as a function of β or γ:



Appendix: Limitations on the value of E_{max}

There are various limits to the maximum value of E_{max} in the calculation of the energy loss during a single collision between projectile and target particles. These can be grouped as

I) Cases where the above calculation is no longer valid

II) Absolute limits on the transferable energy.

A proper relativistic and quantum mechanical treatment is needed for an accurate result, but the following arguments give a reasonable approximation to the limiting value for E_{max}, or the corresponding quantity b_{min}.

I a) The above treatment assumes a non-relativistic expression can be used for the energy of the scattered electron. This implies $E_e < m_e c^2$.
From equation [2], the impact parameter is therefore limited to

$$b_{min}^{(1)} = \frac{\sqrt{2}\, z\, e^2}{\beta\, 4\pi\varepsilon_0\, m_e c^2} \ .$$

I b) The derivation is also only valid when the motion of the target electron is negligible during the collision. This is less restrictive than a) for a relativistic incident particle.

II a) The maximum energy transferable during the collision is limited by conservation of energy and momentum. It can be shown (see the problem!) that

$$\Delta E_{max} = \frac{2\gamma^2 \beta^2\, m_e c^2}{1 + 2\gamma \dfrac{m_e}{M} + \left(\dfrac{m_e}{M}\right)^2} \qquad [7]$$

Note 1) For p, π or μ, $\left(\frac{m_e}{M}\right)^2 << 1$, so the last term of the denominator can be neglected.

2) In the <u>extreme</u> relativistic case, $\gamma >> \frac{M}{m_e}$ (i.e. a proton energy >> 1.7 TeV or pion energy >> 38 GeV!!), then

$$\Delta E_{max} = \frac{2\gamma^2 \beta^2\, m_e c^2}{2\gamma \dfrac{m_e}{M}} = \gamma\beta^2 Mc^2 \approx E$$

In other words, almost all the energy can be lost in a single collision.

3) In the normal case, $\gamma << \frac{M}{m_e}$, then

$$\Delta E_{max} = 2\gamma^2 \beta^2\, m_e c^2$$

(i.e. independent of M, the mass of the projectile particle).

4) For incoming electrons, $M = m_e$, so

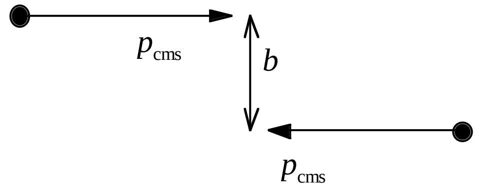
$$[7] \Rightarrow \qquad \Delta E_{max} = \frac{\gamma^2 \beta^2\, m_e c^2}{1+\gamma} \longrightarrow E \quad \text{at large } \gamma .$$

So, in summary, for an incoming p, π or μ at most energies, the maximum energy transfer is $E_{max} = 2\gamma^2 \beta^2\, m_e c^2$.

Again, for a relativistic incoming particle, this is not as restrictive as the limit calculated in Ia).

II b) Quantum mechanical uncertainty in b

Consider the collision in the centre of mass frame:



Angular momentum $L = p_{\rm cms}\, b$ but $\delta L\, \delta\phi \approx \hbar$

$$\Rightarrow \delta L \approx \hbar \quad \text{so} \quad \delta b = \hbar / p_{\rm cms}.$$

For a heavy incident particle, $p_{\rm cms} \approx \dfrac{m_e p}{M}$

$$\Rightarrow \qquad b_{\rm min}^{\rm q.m.} \approx \frac{\hbar M}{m_e p} = \frac{\hbar}{m_e \gamma \beta c}$$

$$\left[\text{For an incoming electron, } p_{\rm cms} = \sqrt{\frac{m_e c p}{2}} \quad \Rightarrow \quad b_{\rm min}^{\rm q.m.} \approx \sqrt{\frac{2}{\gamma}}\, \frac{\hbar}{m_e c} \right]$$

We now need to determine which is the most restrictive limit.

$$\frac{b_{\rm min}^{\rm q.m.}}{b_{\rm min}^{(1)}} = \frac{\dfrac{\hbar}{m_e \gamma \beta c}}{\dfrac{\sqrt{2} z e^2}{\beta\, 4\pi\varepsilon_0\, m_e c^2}} = \frac{\hbar c\, 4\pi\varepsilon_0}{\sqrt{2}\gamma\, z e^2} = \frac{137}{\sqrt{2}\gamma\, z}$$

i.e. The quantum mechanical limit is most restrictive until γ is <u>very</u> large.