Yukawa Potential and the Propagator Term

Consider the electrostatic potential about a charged point particle. This is given by $\nabla^2\phi = 0$, which has the solution $\phi = \frac{e}{4\pi\varepsilon_0 r}$. This describes the potential for a force mediated by massless particles, the photons.

For a particle with mass, the relativistic equation $E^2 = p^2c^2 + m^2c^4$ can be converted into a wave equation by the substitutions

$$E \to i\hbar\frac{\partial}{\partial t}; \quad p_x \to -i\hbar\frac{\partial}{\partial x} \quad \text{etc.}$$

Hence,
$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = \left(m^2c^4 - \hbar^2c^2\nabla^2\right)\phi \ .$$

Or, in the static, time independent case, this leads to

$$\left(\nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0\,,$$

(which gives $\nabla^2\phi = 0$ for the massless case, as required).

For a point source with spherical symmetry, the differential operator can be written as

$$\nabla^2\phi \to \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) \equiv \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi)\,, \qquad \text{so} \qquad \frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi) = \frac{m^2c^2}{\hbar^2}r\phi$$

with solution

$$\phi = g^2\frac{e^{-r/R}}{r}$$

where g is a constant (the coupling strength) and $R = \hbar/mc$ is the range of the force. This is known as the Yukawa form of the potential, and was originally introduced to describe the nuclear interaction between protons and neutrons due to pion exchange.

Using this form of potential and the Born approximation leads, after some manipulation (see the homework!), to a matrix element given by

$$M_{\mathrm{fi}} = \frac{4\pi g^2\hbar^2}{q^2 + m^2c^2} \ .$$

Returning to our normal convention of setting $c = 1$, the terms in the denominator give $\frac{1}{q^2 + m^2}$ and this is called the propagator term. It arises from the exchange of a virtual boson whose rest mass (as a physical particle) is m.

The cross-section is proportional to $\left|M_{\mathrm{fi}}\right|^2 \propto \frac{1}{\left(q^2 + m^2\right)^2}$.

Invariance Principles and Conservation Laws

Without invariance principles, there would be no laws of physics! We rely on the results of experiments remaining the same from day to day and place to place. An invariance principle reflects a basic symmetry, and is always intimately related to a conservation law (and to a quantity that cannot be determined absolutely). The proofs presented in the lectures rely on some key results from quantum mechanics.

It is not necessary to be able to reproduce the following proofs, but you must be able to use the results.

First, operators corresponding to physical observables are Hermitian. That is, they obey $\left(\hat{A}\psi\right)^* = \psi^* \hat{A}$ for any wave-function ψ. The proof is as follows.

The expectation value of an observable $\langle a\rangle = \int \psi^* \hat{A}\psi \, \mathrm{d}^3\mathbf{r}$ must be real.

$$\Rightarrow \quad \int \psi^* \left(\hat{A}\psi\right) \mathrm{d}^3\mathbf{r} = \int \psi \left(\hat{A}\psi\right)^* \mathrm{d}^3\mathbf{r} \qquad (1)$$

We can expand ψ in terms of another set of wave-functions, ϕ_i. $\psi = \sum_i c_i \phi_i$. Substitute this into (1).

$$\Rightarrow \quad \int \sum_i c_i^* \phi_i^* \left(\hat{A} \sum_j c_j \phi_j \right) \mathrm{d}^3\mathbf{r} = \int \sum_j c_j \phi_j \left(\hat{A} \sum_i c_i \phi_i \right)^* \mathrm{d}^3\mathbf{r}$$

$$\Rightarrow \quad \sum_i \sum_j c_i^* c_j \int \phi_i^* \hat{A}\phi_j \, \mathrm{d}^3\mathbf{r} = \sum_j \sum_i c_j c_i^* \int \phi_j \left(\hat{A}\phi_i\right)^* \mathrm{d}^3\mathbf{r}$$

But c_i and c_j are arbitrary, so $\int \phi_i^* \hat{A}\phi_j \, \mathrm{d}^3\mathbf{r} = \int \phi_j \left(\hat{A}\phi_i\right)^* \mathrm{d}^3\mathbf{r}$,

and as this is for any ϕ_j, $\phi_i^* \hat{A} = \left(\hat{A}\phi_i\right)^*$, as required.

Next, we use this identity to show the important result that **operators which commute with the Hamiltonian** (the operator for total energy) $\hat{H}$ **correspond to quantities which are conserved,** that is **they are "constants of the motion"**.

Consider the time-dependence of $\langle a\rangle = \int \psi^* \hat{A}\psi \, \mathrm{d}^3\mathbf{r}$ where we assume that $\hat{A}$ does not depend explicitly on time *t*. (i.e. $\hat{A}$ is not $\hat{A}(t)$.)

Then
$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = \int \left(\frac{\mathrm{d}\psi^*}{\mathrm{d}t} \hat{A}\psi + \psi^* \hat{A} \frac{\mathrm{d}\psi}{\mathrm{d}t} \right) \mathrm{d}^3\mathbf{r} .$$

We make use of the time-dependent Schrodinger's equation: $\hat{H}\psi = i\hbar \frac{\partial\psi}{\partial t}$.

Hence $\frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}\hat{H}\psi$ with its complex conjugate $\frac{\partial\psi^*}{\partial t} = \frac{i}{\hbar}\left(\hat{H}\psi\right)^*$.

But $\hat{H}$ is a Hermitian operator, so from the above result $\frac{\partial\psi^*}{\partial t} = \frac{i}{\hbar}\psi^* \hat{H}$.

Hence $\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = \int \left(\frac{i}{\hbar}\psi^* \hat{H}\hat{A}\psi - \frac{i}{\hbar}\psi^* \hat{A}\hat{H}\psi \right) \mathrm{d}^3\mathbf{r} = \frac{i}{\hbar}\int \left(\psi^* \left[\hat{H},\hat{A}\right]\psi \right) \mathrm{d}^3\mathbf{r} = 0$ if $\hat{H}$ and $\hat{A}$ commute.

Some classical invariance principles are related to the nature of space-time. Invariance of the Hamiltonian (the operator or expression for total energy) under a translation for an isolated, multiparticle system leads directly to the conservation of the total momentum of the system. This can be demonstrated classically, but we will take a quantum mechanical approach, defining an operator $\hat{D}$ which produces a translation of the wavefunction through δx:

$$\hat{D}\,\psi(x) = \psi'(x) = \psi(x + \delta x)\,.$$

Then it can be shown that $\hat{D} \equiv \exp(i\,\hat{P}\,\delta x / \hbar)$ where $\hat{P}$ is the momentum operator.

$\hat{P}$ is said to act as a "generator of translations". Now since the energy of an isolated system cannot be affected by a translation of the whole system, $\hat{D}$ must commute with the Hamiltonian operator $\hat{H}$, i.e. $[\hat{D}, \hat{H}] = 0$; it must therefore also be true that $[\hat{P}, \hat{H}] = 0$, and so $\hat{P}$ has eigenvalues which are constants of the motion.

We therefore have three equivalent statements:

i) The Hamiltonian is invariant under spatial translations. (Equivalently, it is impossible to determine absolute positions.)

ii) The momentum operator commutes with the Hamiltonian.

iii) Momentum is conserved in an isolated system.

Another conserved quantity is **electric charge**, corresponding to an invariance of physical systems under a translation in the electrostatic potential. In classical electrostatics, absolute potential is arbitrary – the physics only depends on potential *differences*. Assuming this fact remains true, we can consider what would be the consequences of the possibility of creating and destroying electric charge. By hypothesis, the energy required to create a charge Q at a potential Φ_1 would be W, independent of Φ. But we could move the charge to another point at Φ_2, liberating an energy $(\Phi_1 - \Phi_2)Q$, before destroying it with an energy release W: a net energy gain of $(\Phi_1 - \Phi_2)Q$. The ability to create or destroy charge thus violates conservation of energy. Inverting the argument, conservation of energy together with invariance with respect to a change in electric potential automatically requires charge to be conserved. Again, an invariance principle implies a conservation law.

Quantum mechanically, we may define a charge operator $\hat{Q}$ which, when it operates on a wavefunction ψ_q describing a system of total charge q, returns an eigenvalue of q.

$$\hat{Q}\;\psi_q = q\,\psi_q$$

If q is conserved, $\hat{Q}$ and $\hat{H}$ must commute, and (as will be shown in the lecture) this is assured by invariance under a global phase (or gauge) transformation

$$\psi'_q = \exp(i\,\varepsilon\,\hat{Q})\,\psi_q$$

where ε is an arbitrary real parameter. This is very closely analogous to the relationship between conservation of momentum and invariance under displacement, as will be demonstrated in the lectures. (Invariance under a *local* gauge transformation, where ε can depend on space and time, has much greater consequences, but that is beyond the scope of this lecture course.)

The above continuous transformations led to additive conservation laws – the sum of all charges or momenta is conserved. There are also discrete or discontinuous transformations, which lead to multiplicative conservation laws. An important group of these are parity P, charge conjugation C and time reversal T.

The parity operator inverts spatial coordinates. It therefore transforms $\underline{x}$ into $-\underline{x}$, $\underline{p}$ into $-\underline{p}$ etc. In other words, polar vectors change sign; axial vectors, such as angular momentum $\underline{J}$, do not.

Now $$\mathrm{P}\,\psi(\underline{x}) = \psi(-\underline{x}).$$
$$\mathrm{P}^2\,\psi(\underline{x}) = \mathrm{P}\,\psi(-\underline{x}) = \psi(\underline{x}).$$
The parity operator thus has eigenvalues of ±1.

If $\mathrm{P}\,\psi(\underline{x}) = +\psi(\underline{x})$ the wavefunction is said to have *even* parity,
while if $\mathrm{P}\,\psi(\underline{x}) = -\psi(\underline{x})$ it has *odd* parity.
(Note that wavefunctions do not have to be eigenfunctions of parity.)

The spherical harmonics $Y_l^m(\theta, \phi)$ (met in atomic physics and elsewhere) are examples of eigenfunctions of the parity operator. (If they are not familiar, look them up!) By considering a reflection in the origin, it should be clear that in spherical polar coordinates, the parity operator causes
$$r \rightarrow r \quad \text{(unchanged)}$$
$$\theta \rightarrow \pi - \theta$$
$$\phi \rightarrow \pi + \phi\,,$$
and by inspection of the form of the spherical harmonics it can be seen that Y_l^m changes sign if l is odd and remains the same if it is even,
i.e. $$\mathrm{P}\,Y_l^m = (-1)^l\,Y_l^m\,.$$

Invariance with respect to P leads to multiplicative conservation laws.
E.g. Consider $\mathrm{a} + \mathrm{b} \rightarrow \mathrm{c} + \mathrm{d}$
The initial state wavefunction can be written as $\psi_\mathrm{i} = \psi_\mathrm{a}\,\psi_\mathrm{b}\,\psi_l$, where ψ_a and ψ_b are "internal" wavefunctions for particles a and b, and ψ_l is the wavefunction describing their relative motion which depends on the angular momentum l. The P operator affects each factor, so $\mathrm{P}\psi_\mathrm{i} = \mathrm{P}\psi_\mathrm{a}\,\mathrm{P}\psi_\mathrm{b}\,\mathrm{P}\psi_l$
If the intrinsic parities of the particles are given by $\mathrm{P}\psi_\mathrm{a} = \pi_\mathrm{a}\,\psi_\mathrm{a}$, etc., where π_a is just a number ±1, then
$$\mathrm{P}\psi_\mathrm{i} = \pi_\mathrm{a}\,\pi_\mathrm{b}\,(-1)^l\,\psi_\mathrm{i} \quad \text{or} \quad \pi_\mathrm{i} = \pi_\mathrm{a}\,\pi_\mathrm{b}\,(-1)^l$$
i.e. the parity of a multiparticle system is given by the product of the intrinsic parities of the individual particles and the parity of the wavefunction describing their relative motion.

A similar expression can be written for the final state. Thus, if the interaction responsible for the above process is invariant under parity (as the electromagnetic interaction is) then
$$\pi_\mathrm{a}\,\pi_\mathrm{b}\,(-1)^l = \pi_\mathrm{c}\,\pi_\mathrm{d}\,(-1)^{l'}$$ where l' is the final relative angular momentum – the allowed (change in) angular momentum depends on the intrinsic parities of initial and final state particles.

Another discrete transformation is charge conjugation, C, which changes a particle into its antiparticle. This reverses the charge, magnetic moment, baryon number and lepton number of the particle.

Time reversal, T, reverses the time coordinate. However, as will be shown in the lectures, T does not satisfy the simple eigenvalue equation
$$\mathrm{T}\,\psi(t) = \psi(-t) = a\,\psi(t).$$
(In fact, defining T like this not only does not have the desired effect of causing momentum to be reversed while leaving energy unchanged, it results in a wavefunction which does not obey Schrodinger's equation.)

Instead T must be defined by $$\mathrm{T}\,\psi(t) = \psi^*(-t).$$

The strong and electromagnetic interactions are invariant under C, P and T transformations. This is not true of the weak interaction, as can be seen by considering neutrinos (which are only involved in weak interactions). It is observed that neutrinos are always “left-handed”, i.e. their spin is antiparallel to their direction of motion. The P operator reverses momentum but not spin, so when applied to a neutrino would produce a right-handed neutrino, which is not observed. Similarly C applied to a neutrino produces an unobserved left-handed antineutrino. Weak interactions therefore violate C and P. The combination CP, however, applied to a left-handed neutrino produces a right-handed antineutrino, which is observed. Therefore (to a good approximation) weak interactions are invariant under the combined transformation CP. The weak interaction, and all other interactions, are exactly invariant under the combination CPT.

<u>Summary</u>

<u>Invariance</u>	<u>Conserved Quantity</u>
Gravitation, weak, electromagnetic and strong interactions are independent of:	
translation in space	linear momentum
rotations in space	angular momentum
translations in time	energy
EM gauge transformation	electric charge
CPT	(product of parities below)
Gravitation, electromagnetic and strong interactions are independent of:	
spatial inversion P	spatial parity
charge conjugation C	“charge parity”
time reversal T	“time parity”