Scattering and Form Factors

Consider the scattering of an electron by a nucleus (as will be discussed in the Nuclear Physics course in the determination of nuclear size) If the electron has a low energy (compared with the separation of nuclear energy levels) the nucleus is left unexcited, and the scattering is elastic.

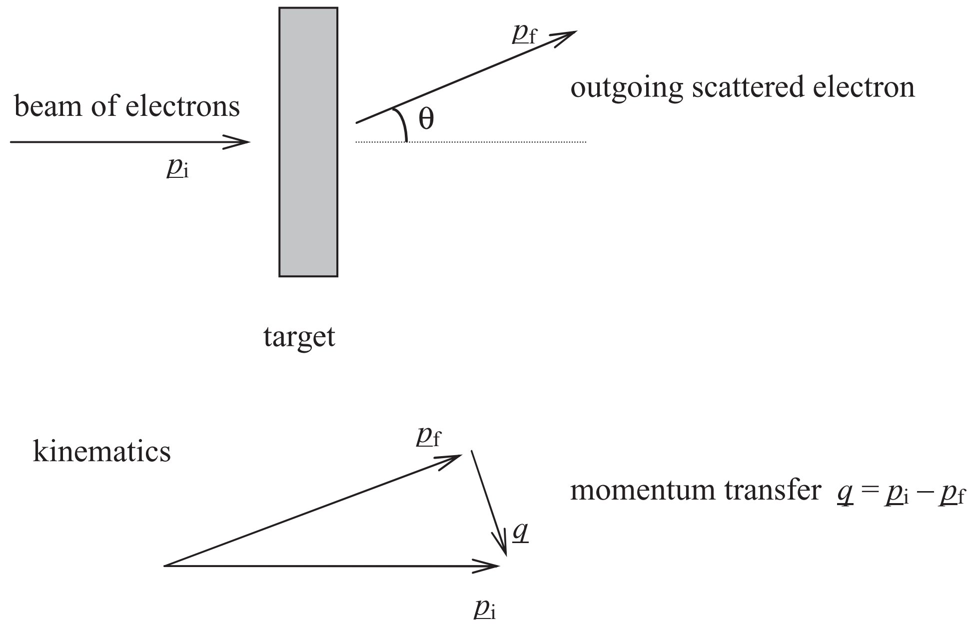

Fig. 1 Definition of momentum transfer in a scattering event.

For elastic scattering $|\underline{p}_i| = |\underline{p}_f| = p$.

At low energies, the centre of mass frame is almost the same as the laboratory frame, since the mass of the electron is very much less than that of the nucleus. (Note that this is only true for low energy electrons.) Then $q^2 = (\underline{p}_i - \underline{p}_f)^2 = p_i^2 + p_f^2 - 2\,\underline{p}_i \cdot \underline{p}_f = 2\,p^2\,(1 - \cos\theta)$

[$\Rightarrow q = 2\,p \sin(\theta/2) \approx p\,\theta$, though this is not important for this discussion.]

Now let us consider the probability of scattering into a given region of solid angle, $d\Omega$. This is $d\sigma/d\Omega$. As described in the introductory notes, such a transition rate is given by Fermi's Golden Rule

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}|\mathbf{M}_{fi}|^2 D_f$$

where $\mathbf{M}_{fi}$ is the scattering amplitude or matrix element containing the dynamics of the interaction, and D_f is the density of final states or phase space factor. (A transition is more likely to occur if the system has more allowed states into which it can move.) A process where the differential cross section is dominated by the density of states factor is nuclear beta decay, discussed in depth in the nuclear physics course, and this factor will not be considered further here. Instead we return to the low energy elastic scattering case introduced above, which is dominated by the scattering amplitude.

For a spin-less electron scattering from a point nuclear charge, $d\sigma/d\Omega$ is given by the classical Rutherford scattering cross section. In reality, the electron has spin $\frac{1}{2}\hbar$, and the form is known as the Mott formula, but we will not pursue this. Instead we will consider the effect on spin-less electrons of having a finite sized nuclear charge. $\mathbf{M}_{fi}$ can be calculated using the "Born approximation" – this assumes that a single scattering occurs, and that the initial and final state electrons can be described by plane waves. We will also ignore the recoil of the nucleus, which we have already indicated is a reasonable approximation in the low energy limit, especially for a heavy nucleus.

The scattering amplitude (or matrix element) is then given by $\mathbf{M}_{\mathrm{fi}} = \int_{\text{space}} \psi_{\mathrm{f}}^{*}\, V(\underline{r})\, \psi_{\mathrm{i}}\; \mathrm{d}^3\underline{r}$, where $\mathrm{d}^3\underline{r}$ represents a volume element. Using plane wave functions, this can then be written as

$$\mathbf{M}_{\mathrm{fi}} = \int e^{-i\underline{p}_f\cdot\underline{r}/\hbar}\; V(\underline{r})\, e^{i\underline{p}_{\mathrm{i}}\cdot\underline{r}/\hbar}\; \mathrm{d}^3\underline{r} = \int e^{i\underline{q}\cdot\underline{r}/\hbar}\; V(\underline{r})\,\mathrm{d}^3\underline{r} \qquad \text{(A)}$$

If we express the nuclear charge density as $Z\,e\,\rho(\underline{r}')$, with $\int\rho(\underline{r}')\,\mathrm{d}^3\underline{r}' = 1$, then the potential energy of the electron at $\underline{r}$ is

$$V(\underline{r}) = -\frac{Ze^2}{4\pi\varepsilon_0}\int\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\mathrm{d}^3\underline{r}'$$ where the integral is over all the nuclear charge.

So (A) becomes $\mathbf{M}_{\mathrm{fi}} = -\dfrac{Ze^2}{4\pi\varepsilon_0}\displaystyle\int e^{i\underline{q}\cdot\underline{r}/\hbar}\int\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\,\mathrm{d}^3\underline{r}'\,\mathrm{d}^3\underline{r}$.

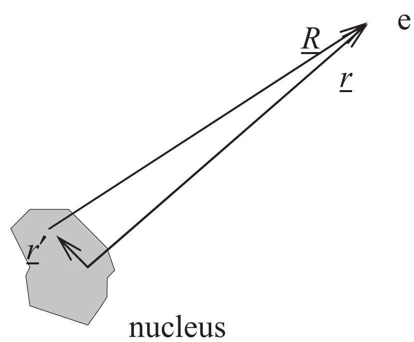

Fig. 2 Definition of $\underline{r}$, $\underline{r}'$ and $\underline{R}$.

To simplify this, we can write $\underline{R} = \underline{r} - \underline{r}'$, and note that for a given $\underline{r}'$, $\mathrm{d}^3\underline{R} = \mathrm{d}^3\underline{r}$. Reorganising then gives us

$$\mathbf{M}_{\mathrm{fi}} = -\frac{Ze^2}{4\pi\varepsilon_0}\int e^{i\underline{q}\cdot\underline{R}/\hbar}\left[\int\frac{e^{i\underline{q}\cdot\underline{r}'/\hbar}\,\rho(\underline{r}')}{|\underline{R}|}\,\mathrm{d}^3\underline{r}'\right]\mathrm{d}^3\underline{R}$$

$$= -\frac{Ze^2}{4\pi\varepsilon_0}\int\frac{e^{i\underline{q}\cdot\underline{R}/\hbar}}{|\underline{R}|}\,\mathrm{d}^3\underline{R}\quad \underline{\underline{\int e^{i\underline{q}\cdot\underline{r}'/\hbar}\,\rho(\underline{r}')\,\mathrm{d}^3\underline{r}'}}\;.$$

We can now consider some special cases:

i) For a point-like nucleus, the charge is a δ-function at $\underline{r}' = 0$, and the term indicated $\underline{\underline{\int e^{i\underline{q}\cdot\underline{r}'/\hbar}\,\rho(\underline{r}')\,\mathrm{d}^3\underline{r}'}} = 1$. We now have Rutherford scattering.

The modification due to the finite size of the nucleus is known as the <u>form factor</u>, $F(\underline{q}) = \int e^{i\underline{q}\cdot\underline{r}'/\hbar}\,\rho(\underline{r}')\,\mathrm{d}^3\underline{r}'$. It can be seen that this is just the Fourier transform of the charge distribution.

NOTE: The Fourier relationship between scattered amplitude and spatial distribution of the scatterer is general, e.g. optical diffraction, X-ray scattering, etc.

ii) Spherically symmetric charge distribution $\rho(\underline{r}') = \rho(r')$. We can choose spherical co-ordinates as shown in figure 3, with the z-axis parallel to $\underline{q}$. Then the volume element $\mathrm{d}^3\underline{r}' = r'^2\,\mathrm{d}r'\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi$.

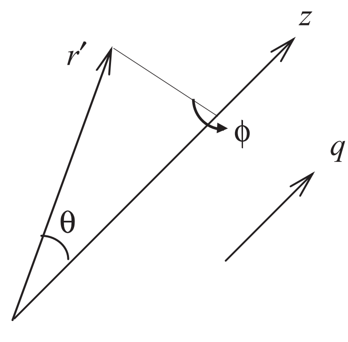

Fig. 3 Choice of coordinates.

In the spin-less case, we must have symmetry in φ, so integrating we obtain

$$F(\underline{q}) = \int_{r'}\int_{-1}^{1}\int_{0}^{2\pi}\rho(r')\,e^{iqr'\cos\theta/\hbar}\,r'^2\,\mathrm{d}\phi\,\mathrm{d}\cos\theta\,\mathrm{d}r'$$

$$= \int_{r'}\int_{-1}^{1}\rho(r')\,e^{iqr'\cos\theta/\hbar}\,2\pi r'^2\,\mathrm{d}\cos\theta\,\mathrm{d}r'$$

$$= \int_{r'}\left[\frac{\hbar}{iqr'}e^{iqr'\cos\theta/\hbar}\right]_{-1}^{1}\rho(r')\,2\pi r'^2\,\mathrm{d}r'$$

$$= \int_{r'} 4\pi\rho(r')\frac{\sin(qr'/\hbar)}{qr'/\hbar}\,r'^2\,\mathrm{d}r'$$

- In principle, the measured ${d\sigma}/{d\Omega}$ can be used to determine $F(\underline{q})$, and then the inverse Fourier transform used to obtain $\rho(\underline{r})$.

  $$\text{i.e.}\quad \rho(r) = \frac{1}{(2\pi)^3}\int F(\underline{q})\, e^{-i\underline{q}\cdot\underline{r}/\hbar}\, \mathrm{d}^3\underline{q}$$

  However, to do this requires knowledge of $F(\underline{q})$ over the complete range of $\underline{q}$, which is impractical (at large q, σ is very small and difficult to determine accurately). In practice, a model for $\rho(r)$ is assumed, described by a small number of parameters, which are then adjusted to best fit the measured values of $F(\underline{q})$.

- Note (from the Fourier relationship) that a broad spatial distribution leads to a narrow distribution in $\underline{q}$. E.g. in the Rutherford experiment, large atoms $\rightarrow$ small scattering displacements
  small nuclei $\rightarrow$ large (but rare) displacements.

<u>Summary</u>

Scattering amplitude due to distributed charge = (scattering amplitude due to point) $\times$ (form factor)

and

Form Factor = Fourier transform of charge distribution.