Hermitian Operators and "Constants of the Motion"

It is not necessary to be able to reproduce the following proofs, but you must be able to use the results.

First, operators corresponding to physical observables are Hermitian. That is, they obey $\left(\hat{A}\psi\right)^* = \psi^* \hat{A}$ for any wave-function ψ. The proof is as follows.

The expectation value of an observable $\langle a \rangle = \int \psi^* \hat{A}\psi \, \mathrm{d}^3\mathbf{r}$ must be real.

$$\Rightarrow \quad \int \psi^* \left(\hat{A}\psi\right) \mathrm{d}^3\mathbf{r} = \int \psi \left(\hat{A}\psi\right)^* \mathrm{d}^3\mathbf{r} \qquad (1)$$

We can expand ψ in terms of another set of wave-functions, ϕ_i. $\psi = \sum_i c_i \phi_i$. Substitute this into (1).

$$\Rightarrow \quad \int \sum_i c_i^* \phi_i^* \left(\hat{A} \sum_j c_j \phi_j \right) \mathrm{d}^3\mathbf{r} = \int \sum_j c_j \phi_j \left(\hat{A} \sum_i c_i \phi_i \right)^* \mathrm{d}^3\mathbf{r}$$

$$\Rightarrow \quad \sum_i \sum_j c_i^* c_j \int \phi_i^* \hat{A} \phi_j \, \mathrm{d}^3\mathbf{r} = \sum_j \sum_i c_j c_i^* \int \phi_j \left(\hat{A} \phi_i \right)^* \mathrm{d}^3\mathbf{r}$$

But c_i and c_j are arbitrary, so $\int \phi_i^* \hat{A} \phi_j \, \mathrm{d}^3\mathbf{r} = \int \phi_j \left(\hat{A} \phi_i \right)^* \mathrm{d}^3\mathbf{r}$,

and as this is for any ϕ_j, $\phi_i^* \hat{A} = \left(\hat{A} \phi_i \right)^*$, as required.

Next, we use this identity to show the important result that **operators which commute with the Hamiltonian** (the operator for total energy) $\hat{H}$ **correspond to quantities which are conserved,** that is **they are "constants of the motion"**.

Consider the time-dependence of $\langle a \rangle = \int \psi^* \hat{A}\psi \, \mathrm{d}^3\mathbf{r}$ where we assume that $\hat{A}$ does not depend explicitly on time t. (i.e. $\hat{A}$ is not $\hat{A}(t)$.)

Then $$\frac{\mathrm{d}\langle a \rangle}{\mathrm{d}t} = \int \left(\frac{\mathrm{d}\psi^*}{\mathrm{d}t} \hat{A}\psi + \psi^* \hat{A} \frac{\mathrm{d}\psi}{\mathrm{d}t} \right) \mathrm{d}^3\mathbf{r} .$$

We make use of the time-dependent Schrodinger's equation: $\hat{H}\psi = i\hbar \dfrac{\partial \psi}{\partial t}$.

Hence $\dfrac{\partial \psi}{\partial t} = \dfrac{-i}{\hbar} \hat{H}\psi$ with its complex conjugate $\dfrac{\partial \psi^*}{\partial t} = \dfrac{i}{\hbar} \left(\hat{H}\psi \right)^*$.

But $\hat{H}$ is a Hermitian operator, so from the above result $\dfrac{\partial \psi^*}{\partial t} = \dfrac{i}{\hbar} \psi^* \hat{H}$.

Hence $\dfrac{\mathrm{d}\langle a \rangle}{\mathrm{d}t} = \int \left(\dfrac{i}{\hbar} \psi^* \hat{H}\hat{A}\psi - \dfrac{i}{\hbar} \psi^* \hat{A}\hat{H}\psi \right) \mathrm{d}^3\mathbf{r} = \dfrac{i}{\hbar} \int \left(\psi^* \left[\hat{H}, \hat{A} \right] \psi \right) \mathrm{d}^3\mathbf{r} = 0$ if $\hat{H}$ and $\hat{A}$ commute.