Fermions and Bosons

A multi-particle wave function for non-interacting (e.g. widely separated) particles can be written as the product of single particle functions.

$$\Psi(1,2,3,...) = c\ \psi_A(1)\psi_B(2)\psi_C(3)...$$

where A, B, C describe the quantum numbers of the state and 1, 2, 3 give the co-ordinates of the particle. (c is simply a normalisation constant.) Observables are given by the square of the wave function $|\Psi|^2$.

If we consider a number of identical, indistinguishable particles, then clearly interchanging a pair of particles is quite unobservable.

$$|\Psi(2,1,3)|^2 = |\Psi(1,2,3)|^2$$

This has two possible solutions

$$\Psi(2,1,3) = +\Psi(1,2,3) \qquad (1)$$
$$\Psi(2,1,3) = -\Psi(1,2,3) \qquad (2)$$

These two cases have important physical consequences. E.g. if two particles are in identical quantum states

$$\Psi(1,2) = c\ \psi_A(1)\psi_A(2)$$

case (1) implies $c_1\ \psi_A(2)\psi_A(1) = c_1\ \psi_A(1)\psi_A(2)$ which is clearly satisfied by any c_1,
case (2) implies $c_2\ \psi_A(2)\psi_A(1) = -c_2\ \psi_A(1)\psi_A(2)$ which is only satisfied if $c_2 = 0$ – the wave function is zero.

Particles obeying the two conditions have completely different behaviours.

Bosons have wave functions which are symmetric under the interchange of identical particles. They obey Bose-Einstein statistics, showing constructive interference of identical single particle wave-functions. Writing down a wavefunction which is guaranteed to be symmetric we have

$$\Psi = \tfrac{1}{\sqrt{2}}\left(\psi_A(1)\psi_B(2) + \psi_A(2)\psi_B(1)\right).$$

Fermions have wave functions which are antisymmetric under the interchange of identical particles. They obey Fermi-Dirac statistics, showing destructive interference of identical single particle wave functions. In particular, no two identical fermions can occupy wave functions with identical quantum numbers. The antisymmetric combination is

$$\Psi = \tfrac{1}{\sqrt{2}}\left(\psi_A(1)\psi_B(2) - \psi_A(2)\psi_B(1)\right).$$

Fermions are particles with "half integer" spin, i.e. $\tfrac{1}{2}\hbar$, $\tfrac{3}{2}\hbar$, $\tfrac{5}{2}\hbar$, … (e.g. proton, neutron, electron, neutrino, quarks, …).
They include the constituent particles of matter. For each particle, there is a distinct antiparticle

e.g. $e^- \leftrightarrow e^+$
$\nu_e \leftrightarrow \bar{\nu}_e$ both neutral, but different.

They obey conservation laws – they are only produced as fermion-antifermion pairs.

Bosons have integer spin, i.e. 0, $\hbar$, $2\hbar$, … (e.g. photons, π (pi meson) and other mesons, $W^\pm$, Z, gluon, …)
They include the quanta of fields, i.e. the carriers of forces.
They can be created and destroyed e.g. $e^- + e^- \to e^- + e^- + \gamma$

Bosons can be their own antiparticles e.g.

$$\begin{array}{ccc} \pi^+ & & \pi^- \\ \pi^0 & \leftrightarrow & \pi^0 \\ \pi^- & & \pi^+ \end{array}$$

The <u>fundamental</u> fermions are believed to be the electron-like particles known as <u>leptons</u> and the <u>quarks</u>. (As we will see later, the proton and neutron – examples of <u>baryons</u> – are made of quarks. So are mesons, and together they make up the <u>hadrons</u>.)

The basic constituents are the electron and neutrino (leptons) and u and d quarks. The leptons have an associated lepton number L which is (as far as we know) absolutely conserved.

Leptons e^- and ν have $L = +1$;
Antileptons e^+ and $\bar{\nu}$ have $L = -1$.

In fact, though the above 4 particles are all that is required to make the present day universe, the pattern is repeated, occurring 3 times, with heavier, unstable versions of the ordinary particles. So, in the lepton sector we have:

	e^-	μ^-	τ^-	
mass	0.511	105.7	1777	MeV/c^2
	ν_e	ν_μ	ν_τ	
extremely small	<2 <u>eV</u>/c^2	<2 <u>eV</u>/c^2	<2 <u>eV</u>/c^2	

and their antiparticles.

[The above neutrino mass limits are the result of oscillation experiments; direct measurements in decays give much poorer limits for the mass of ν_μ *and* ν_τ, *0.19 MeV/*c^2 *and 18 MeV/*c^2 *respectively. In all kinematic calculations we will treat the three species of neutrino as massless. Recent evidence indicates that, though these masses are certainly very small, they are not zero. This may have important consequences for our theories of the particle families. For more information on these recent results, please see the web pages – but note this is definitely non-examinable!]*

Each generation has its own distinct lepton number, L_e, L_μ, L_τ. This is what makes the different neutrinos distinct, and forbids $\mu \to e\,\gamma$.

As we will see later, the quark types can be changed by the weak interaction, so there is only a global baryon number B. Quarks have $B = +\frac{1}{3}$, antiquarks have $B = -\frac{1}{3}$.

E.g. proton (uud) has $B = +1$; neutron (udd) has $B = +1$;
$\bar{p}\left(\bar{u}\bar{u}\bar{d}\right)$ has $B = -1$; $\pi^+ (u\bar{d})$ has $B = 0$.

We will discuss the heavier quarks, and the allowed combinations of quarks which form hadrons, later.

Specimen Kinematic Calculations

A) **Inelastic Scattering**
Electrons of energy 900 MeV scatter inelastically off protons. If an electron scattered through 30º has an energy of 234 MeV, what is the mass of the "excited proton" it creates?

B) **Two-Body Decay**
i) Massless particles
A π^0 (with mass 135 MeV/c^2) at rest decays into 2 photons, $\pi^0 \to \gamma\gamma$.
What are their energies and momenta?

ii) Particles with mass
A ρ^0 meson (mass 770 MeV/c^2) at rest decays into 2 charged pions (mass 140 MeV/c^2) according to $\rho^0 \to \pi^+\pi^-$.
What are their energies and momenta?

C) **Two-body Collision**
What energy pions must strike a proton at rest in order to produce a Δ baryon (of mass 1232 MeV/c^2)?