Invariance Principles & Conservation Laws

Without invariance principles, there would be no laws of physics! We rely on the results of experiments remaining the same from day to day and place to place. An invariance principle reflects a basic symmetry, and is always intimately related to a conservation law (and to a quantity that cannot be determined absolutely).

Some invariance principles are related to the nature of space-time. Invariance of the Hamiltonian (the operator or expression for total energy) under a translation for an isolated, multi-particle system leads directly to the conservation of the total momentum of the system. This can be demonstrated classically, but we will take a quantum mechanical approach:

Define an operator $\hat{D}$ which produces a translation of the wavefunction through δx:

$$\hat{D}\psi(x) = \psi'(x) = \psi(x+\delta x)$$

→ Then it can be shown that

$$\hat{D} = \exp\left(\frac{i}{\hbar}\hat{P}\,\delta x\right)$$

where $\hat{P}$ is the momentum operator.

$\hat{P}$ is said to act as a "generator of translations".

Now since the energy of an isolated system cannot be affected by a translation of the whole system, $\hat{D}$ must

→ commute with the Hamiltonian operator , i.e. $\left[\hat{D},\hat{H}\right]=0$;

it must therefore also be true that $\left[\hat{P},\hat{H}\right]=0$, and so $\hat{P}$ also has eigenvalues which are constants of the motion.

We therefore have three equivalent statements:

i) Momentum is conserved in an isolated system.

ii) The Hamiltonian is invariant under spatial translations. (Equivalently, it is impossible to determine absolute positions.)

iii) The momentum operator commutes with the Hamiltonian.

Summary of key point of argument

Invariance Principles & Conservation Laws

Principle

Operator commutes with Hamiltonian $\Rightarrow$ eigenvalues are conserved quantities

i.e. $\left[\hat{O},\hat{H}\right]=0 \quad \Rightarrow \quad \text{If } \hat{O}\psi = o\psi$ then o is conserved.

Application

Physics is unchanged by translation in space $\Rightarrow \left[\hat{D},\hat{H}\right]=0$

But $\hat{P}$ is generator of displacements, as $\hat{D} = \exp\left(\frac{i}{\hbar}\hat{P}\,\delta x\right)$

so $\Rightarrow \left[\hat{P},\hat{H}\right]$ also $=0 \quad \Rightarrow$ momentum is a conserved quantity.

Another conserved quantity is electric charge, corresponding to an invariance of physical systems under a translation in the electrostatic potential – the physics only depends on potential *differences*.

- Consider consequence of creating charge (while this remains true).
- Energy required to create charge Q at potential Φ_1 is W (independent of Φ).
- Move Q to point at potential Φ_2, releasing energy $(\Phi_1 - \Phi_2)Q$
- Destroy charge Q, liberating W again
- Net energy gain is $(\Phi_1 - \Phi_2)Q$
- Creating/destroying charge allows energy non-conservation

Inverting the argument:
Conservation of energy + invariance w.r.t. change in potential
$\Rightarrow$ conservation of charge

Quantum mechanically, we may define a charge operator $\hat{Q}$ which, when it operates on a wavefunction ψ_q describing a system of total charge *q*, returns an eigenvalue of *q*.

$$\hat{Q}\psi_q = q\psi_q$$

What invariance is required to ensure $\left[\hat{Q},\hat{H}\right]=0$, so that charge is conserved?

→ **If *q* is conserved, $\hat{Q}$ and $\hat{H}$ must commute, and (as we will see) this is assured by invariance under a global phase (or gauge) transformation**

$$\psi_q' = \exp\left(i\varepsilon\hat{Q}\right)\psi_q$$

where ε is an arbitrary real parameter.

What you should have learned this week

- Exchanged particles involved in all 4 interactions
- Uncertainty principle and virtual exchange particles
- Feynman diagrams and antiparticles
- Yukawa potential:
  - Relationship between range and mass
  - Propagator term
- Relationship between conserved quantity and invariance
  - Examples of displacement (momentum) and charge

Reading for next time from "Ideas of Particle Physics":

- ***Chapter 6 – Symmetry and CPT***
- ***Chapter 4.7 - 4.10 – Virtual processes & renormalisation***

The previous continuous transformations led to additive conservation laws – the sum of all charges or momenta is conserved.

There are also discrete or discontinuous transformations, which lead to multiplicative conservation laws.

An important group of these are parity P, charge conjugation C and time reversal T.

The parity operator inverts spatial coordinates.
It transforms $\underline{r}$ into $-\underline{r}$, $\underline{p}$ into $-\underline{p}$ etc.

→ Polar vectors change sign; axial vectors, such as angular momentum $\underline{J}$, do not.

Now $P\,\psi(\underline{x}) = \psi(-\underline{x})$.
$P^2\,\psi(\underline{x}) = P\,\psi(-\underline{x}) = \psi(\underline{x})$.
The parity operator thus has eigenvalues of ±1.

If $P\,\psi(\underline{x}) = +\,\psi(\underline{x})$ the wavefunction is said to have *even* parity,
while if $P\,\psi(\underline{x}) = -\,\psi(\underline{x})$ it has *odd* parity.

→ (Note that wavefunctions do not have to be eigenfunctions of parity.)

The spherical harmonics $Y_l^m(\theta,\phi)$ (met in atomic physics and elsewhere) are examples of eigenfunctions of the parity operator. (If they are not familiar, look them up!)

Since, in spherical polar coordinates, the parity operator causes

$$r \rightarrow r$$
$$\theta \rightarrow \pi - \theta$$
$$\phi \rightarrow \pi + \phi$$

by inspection of the form of the spherical harmonics it can be seen that Y_l^m changes sign if l is odd and remains the same if it is even,

i.e. $$P\,Y_l^m = (-1)^l\,Y_l^m\,.$$

[Remember, l is the orbital angular momentum quantum number.]

Invariance with respect to P leads to multiplicative conservation laws.

E.g. Consider $a + b \rightarrow c + d$

The initial state wavefunction can be written as

$$\psi_i = \psi_a\,\psi_b\,\psi_l,$$

where ψ_a and ψ_b are "internal" wavefunctions for particles a and b, and ψ_l is the wavefunction describing their relative motion.

The P operator affects each factor, so $P\psi_i = P\psi_a\,P\psi_b\,P\psi_l$

If the intrinsic parities of the particles are given by $P\psi_a = \pi_a \psi_a$, etc., with $\pi_a = \pm 1$, then

$$P\psi_i = \pi_a \pi_b (-1)^l \psi_i \quad \text{or} \quad \pi_i = \pi_a \pi_b (-1)^l$$

i.e. the parity of a multi-particle system is given by the product of the intrinsic parities of the individual particles and the parity of the wavefunction describing their relative motion.

A similar expression can be written for the final state. Thus, if the interaction responsible for the above process conserves parity (as the strong and electromagnetic interactions do) then

$$\pi_a \pi_b (-1)^l = \pi_c \pi_d (-1)^{l'}$$

where l' is the final relative angular momentum.

Another discrete transformation is charge conjugation, C, which changes a particle into its antiparticle. This reverses the charge, magnetic moment, baryon number and lepton number of the particle.

The strong and electromagnetic interactions are invariant under C, P (and T) transformations.

This is not true of the weak interaction, as can be seen by considering neutrinos (which are only involved in weak interactions).

Neutrinos are always left-handed, i.e. their spin is antiparallel to their direction of motion. The P operator reverses momentum but not spin, so when applied to a neutrino would produce a right-handed neutrino, which is not observed, Similarly C applied to a neutrino produces an unobserved left-handed antineutrino. Weak interactions therefore violate C and P.

The combination CP, however, applied to a left-handed neutrino produces a right-handed antineutrino, which is observed. Therefore (to a good approximation*) weak interactions are invariant under the combined transformation CP.

The weak interaction, and all other interactions, are exactly invariant under the combination CPT.

** - see the web site*

We have considered two discrete transformations:

- Spatial inversion or parity
- Particle-antiparticle inversion, or charge conjugation.

There is a third type of inversion:

Time reversal, T reverses the time coordinate. However, T does not satisfy the simple eigenvalue equation

$$T\,\psi(t) = \psi(-t) = a\,\psi(t).$$

Instead $\quad T\,\psi(t) = \psi^*(-t).$

Summary

Invariance	Conserved Quantity
Gravitation, weak, electromagnetic and *strong interactions* are independent of:	
translation in space	linear momentum
rotations in space	angular momentum
translations in time	energy
EM gauge transformation	electric charge
CPT	(product of parities below)
Gravitation, electromagnetic and *strong interactions* are independent of:	
spatial inversion P	spatial parity
charge conjugation C	"charge parity"
time reversal T	"time parity"

What you should have learned today

- Discrete and continuous transformations
- Additive and multiplicative conservation laws
- Parity
  - $(-1)^l$ multiplicative factor
  - Parity of multiparticle wavefunctions
- Charge conjugation. CP & neutrinos
- Time reversal. $T\,\psi(t) = \psi^*(-t)$.
- Which quantities are conserved by each interaction.