Yukawa Potential and the Propagator Term

Consider the electrostatic potential about a charged point particle.

This is given by $\nabla^2\phi = 0$, which has the solution

$$\phi = \frac{e}{4\pi\varepsilon_0 r}$$

This describes the potential for a force mediated by mass-less particles, the photons

For a particle with mass, the relativistic equation

$$E^2 = p^2c^2 + m^2c^4$$

can be converted into a wave equation by the substitutions

$$E \to i\hbar\frac{\partial}{\partial t}; \qquad p_x \to -i\hbar\frac{\partial}{\partial x} \qquad \text{etc.}$$

Hence,

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = \left(m^2c^4 - \hbar^2c^2\nabla^2\right)\phi$$

Or, in the static, time independent case, this leads to

$$\left(\nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$

(which gives $\nabla^2\phi = 0$ for the massless case, as required).

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For a point source with spherical symmetry, the differential operator can be written as

$$\nabla^2\phi \to \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) \equiv \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi)$$

so $\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi) = \frac{m^2c^2}{\hbar^2}r\phi$ with solution $\phi = g^2\frac{e^{-r/R}}{r}$

where g is a constant (the coupling strength)

and $R = \hbar/mc$ is the range of the force.

This is known as the Yukawa form of the potential, and was originally introduced to describe the nuclear interaction between protons and neutrons due to pion exchange.

Using this form of potential and the Born approximation leads, after some manipulation (see the homework!), to a matrix element given by

$$M_{\mathrm{fi}} = \frac{4\pi g^2\hbar^2}{q^2 + m^2c^2}$$

Setting $c = 1$, the factor involving the denominator $\frac{1}{q^2 + m^2}$ is called the propagator term.

It arises from the exchange of a virtual boson whose rest mass (as a physical particle) is m.

The cross-section is proportional to

$$|M_{\mathrm{fi}}|^2 \propto \frac{1}{\left(q^2 + m^2\right)^2}$$