Elastic Scattering and Form Factors

Warning! Mathematical topic!

Fermi's Golden Rule

Transition rate, or differential cross-section, is given by:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} |\mathbf{M}_{fi}|^2 D_f$$

where $\mathbf{M}_{fi}$ is the scattering amplitude or matrix element containing the dynamics of the interaction, and D_f is the density of final states or phase space factor.

- Spin-less electron scattering from a point nuclear charge → Rutherford scattering cross section
- (Spin ½$\hbar$ electron → Mott formula – not considered here)

Consider effect of finite sized nuclear (or proton) charge (and spin-less electrons):

Use "Born approximation" – assume a single scattering, initial and final state electrons described by plane waves.

Also ignore recoil of nucleus (or proton).

Scattering amplitude (or matrix element) is then given by ,

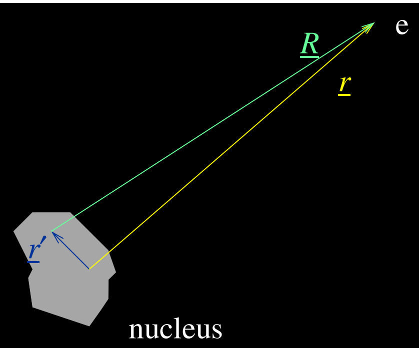
$$\mathbf{M}_{fi} = \int_{\text{space}} \psi_f^* \, V(\underline{r}) \, \psi_i \, d^3\underline{r}$$

where $d^3\underline{r}$ represents a volume element.

Using plane wave functions, this is written as

$$\mathbf{M}_{fi} = \int e^{-i\underline{p}_f \cdot \underline{r}/\hbar} \; V(\underline{r}) \; e^{i\underline{p}_i \cdot \underline{r}/\hbar} \; d^3\underline{r}$$

$$= \int e^{i\underline{q}\cdot\underline{r}/\hbar} \; V(\underline{r}) \; d^3\underline{r} \qquad \text{(A)}$$



Write *normalised* nuclear charge density $Z e \rho(\underline{r}')$, with $\int \rho(\underline{r}') \, d^3\underline{r}' = 1$

then the potential energy of the electron at $\underline{r}$ is

$$V(\underline{r}) = -\frac{Ze^2}{4\pi\varepsilon_0} \int \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} \; d^3\underline{r}'$$

where the integral is over all the nuclear charge.

All space

nucleus

Using (A): $\mathbf{M}_{fi} = \int e^{i\underline{q}\cdot\underline{r}/\hbar}\, V(\underline{r})\ d^3\underline{r}$

$$\Rightarrow \quad \mathbf{M}_{fi} = -\frac{Ze^2}{4\pi\varepsilon_0}\int e^{i\underline{q}\cdot\underline{r}/\hbar}\int \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\, d^3\underline{r}'\, d^3\underline{r}$$

Write $\underline{R} = \underline{r} - \underline{r}'$, note that for a given $\underline{r}'$, $d^3\underline{R} = d^3\underline{r}$

$$\mathbf{M}_{fi} = -\frac{Ze^2}{4\pi\varepsilon_0}\int e^{i\underline{q}\cdot\underline{R}/\hbar}\left[\int \frac{e^{i\underline{q}\cdot\underline{r}'/\hbar}\rho(\underline{r}')}{|\underline{R}|}\, d^3\underline{r}'\right] d^3\underline{R}$$

$$= -\frac{Ze^2}{4\pi\varepsilon_0}\int \frac{e^{i\underline{q}\cdot\underline{R}/\hbar}}{|\underline{R}|}\, d^3\underline{R}\quad \int e^{i\underline{q}\cdot\underline{r}'/\hbar}\rho(\underline{r}')\, d^3\underline{r}'$$

$$\mathbf{M}_{fi} = -\frac{Ze^2}{4\pi\varepsilon_0}\int \frac{e^{i\underline{q}\cdot\underline{R}/\hbar}}{|\underline{R}|}\, d^3\underline{R}\quad \int e^{i\underline{q}\cdot\underline{r}'/\hbar}\rho(\underline{r}')\, d^3\underline{r}'$$

Special cases:

i) Point-like nucleus, charge is a δ-function at $\underline{r}' = 0$,

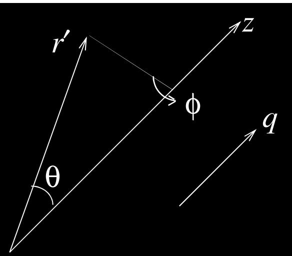
$$\int e^{i\underline{q}\cdot\underline{r}'/\hbar}\rho(\underline{r}')\, d^3\underline{r}' = 1 \quad \text{– Rutherford scattering.}$$

The modification due to the finite size of the nucleus is simply a multiplicative factor known as the form factor,

$$F(\underline{q}) = \int e^{i\underline{q}\cdot\underline{r}'/\hbar}\rho(\underline{r}')\, d^3\underline{r}' \ .$$

This is just the Fourier transform of the charge distribution.

NOTE: The Fourier relationship between scattered amplitude and spatial distribution of the scatterer is general, e.g. optical diffraction, X-ray scattering, etc.



ii) Spherically symmetric charge distribution

$\rho(\underline{r}') = \rho(r')$.

Choose spherical co-ordinates as shown in the figure, with the z-axis parallel to $\underline{q}$. Then the volume element

$d^3\underline{r}' = r'^2\, dr'\, d(\cos\theta)\, d\varphi$.

In the spin-less case, we must have symmetry in φ, so integrating we obtain

$$F(\underline{q}) = \int_{r'}\int_{-1}^{1} \rho(r')\ e^{iqr'\cos\theta/\hbar}\ 2\pi r'^2\ d\cos\theta\ dr'$$

$$= \int_{r'}\left[\frac{\hbar}{iqr'}e^{iqr'\cos\theta/\hbar}\right]_{-1}^{1}\rho(r')\ 2\pi r'^2\ dr'$$

$$= \int_{r'} 4\pi\rho(r')\frac{\sin(qr'/\hbar)}{qr'/\hbar}\ r'^2\ dr'$$

To evaluate further, need to know $\rho(r')$.

In principle, measured $d\sigma/d\Omega$ can be used to determine $F(\underline{q})$, then the inverse Fourier transform used to obtain $\rho(\underline{r})$.

i.e. $$\rho(\underline{r}) = \frac{1}{(2\pi\hbar)^3} \int F(\underline{q})\ e^{-i\underline{q}\cdot\underline{r}/\hbar}\ \mathrm{d}^3\underline{q}$$

However, to do this requires knowledge of $F(\underline{q})$ over the complete range of $\underline{q}$, which is impractical (at large q, σ is very small and difficult to determine accurately). In practice, a model for $\rho(r)$ is assumed, described by a small number of parameters, which are then adjusted to best fit the measured values of $F(\underline{q})$.

Note (from the Fourier relationship) that a broad spatial distribution leads to a narrow distribution in $\underline{q}$.

e.g. Rutherford experiment

large atoms → small scattering displacements

small nuclei → large (but rare) displacements.

What you should have learned this week

- Useful units for E, p, m
- $E^2 = p^2 + m^2$ (in these units)
- Total, partial and differential cross sections
- Momentum transfer $\underline{q} = \underline{p}_i - \underline{p}_f$
- Definition of Form Factor
- Expression for F.F. as FT of charge distribution
- Simplification for spherical symmetry