PHY102 Electricity
Topic 5 (Lectures 7 & 8) – Capacitors and Dielectrics

In this topic, we will cover:
1) Capacitors and Capacitance
2) Combinations of Capacitors – Series and Parallel
3) The energy stored in a capacitor
4) Dielectrics

Reading from Young & Freedman:
For this topic, read the introduction to chapter 24 and sections 24.1 to 24.5.

Introduction

A capacitor is a means of storing electric charge. We saw at the start of the course that an isolated conductor can be charged electrostatically, so this could be considered as the most basic sort of capacitor. The charges on the conductor tend to repel each other, and to reduce their energy they will leak away to ground if any slightly conductive path is available. We can store the charge more easily if we can reduce its energy, which means reducing its potential for a given charge. This can be achieved by arranging that the conductor carrying charge Q is in close proximity to another conductor carrying $-Q$. Although the charges within each conductor may be repelling each other, they will be attracted by those in the other conductor and their net energy will be reduced.

Capacitors and Capacitance

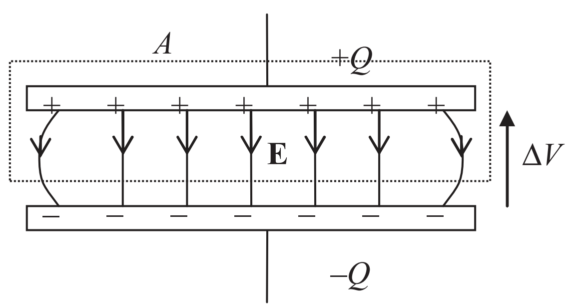
Consider two separated conducting plates, connected to a battery. The electromotive force of the battery drives charge from one terminal to the other. One plate will therefore end up with a positive charge while the other will have an equal negative charge. The potential difference between the plates will be equal to the voltage supplied by the battery. If the battery is disconnected, the charge will remain on the plates, which will maintain the same potential difference. As we will see, the magnitude of the charge stored on either plate Q is directly proportional to the potential difference between the plates. We can write this as

$$\boxed{Q = CV} \qquad [1]$$

where the constant of proportionality C is known as the *capacitance* of the system, which is called a capacitor. The capacitance is thus the quantity of charge stored per unit potential difference between the plates. The S.I. Unit of capacitance is the Farad (F), and 1 Farad = 1 Coulomb / Volt. The Farad (like the Coulomb) is a very large unit, and most practical capacitances are measured in microfarads (1 μF = 10^{-6} F) or picofarads (1 pF = 10^{-12} F). We shall shortly see that capacitors store electrical energy, as well as charge.

Parallel Plate Capacitor

The simplest, and most common, geometry for a capacitor consists of two flat metal plates, close together but separated by a layer of insulating material. When connected to a potential difference, they will carry opposite amounts of electric charge, $\pm Q$. If the separation of the plates is very small compared with their lateral dimensions (as it usually is), we can ignore fringe fields near the ends, and consider a uniform electric field which only exists between the plates.



The attraction between the charges means that they lie on the inner surfaces of the plates. Assume the plates have an area A and separation d. Applying Gauss's law to the dotted surface shown, which completely surrounds one plate, we have

$$EA = \frac{Q}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q}{\varepsilon_0 A}$$

(Note that this is *different* from the case of an isolated sheet of charge considered previously, as here the electric field is all on one side of the sheet, and so only cuts one side of the Gaussian "box".)

We calculate the potential difference V moving from lower to upper plate from

$$V = -\int \mathbf{E} \cdot \mathrm{d}l = Ed$$

where the cancelling minus sign comes from the fact that **E** and the path **d** are in opposite directions.

Thus
$$V = \frac{Qd}{\varepsilon_0 A}.$$

Using our definition for C from [1], we have

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}}. \qquad [2]$$

Rearranging this equation also provides us with an alternative, and more common, unit for ε_0 which is F m^{-1}.

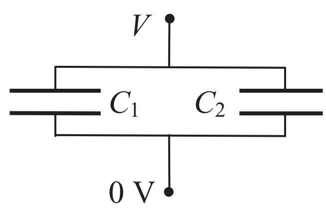
To store a large amount of charge at low potential, i.e. to produce a large capacitance, we obviously want a large plate area A and small separation d. This can be manufactured from two long strips of aluminium foil, separated by a thin sheet of plastic film. In a practical capacitor, this "sandwich" is covered by another sheet of plastic and then rolled up into a cylinder. As we will see later, the plastic layer may have other benefits besides its insulating nature.

<u>Capacitors in Parallel and Series</u>

When two or more capacitors are connected together, they behave like a single capacitor with a modified capacitance. It is useful to be able to calculate the effective capacitance of the combination in terms of the individual capacitances.

Capacitors in Parallel

If two capacitors are connected as shown, with one plate of each connected to ground (0 V) and the other to voltage V, then from [1] capacitor C_1 will carry a charge $Q_1 = C_1 V$ and similarly for C_2. The total charge is $Q = Q_1 + Q_2 = (C_1 + C_2)V = CV$ where C is the effective capacitance of the combination.



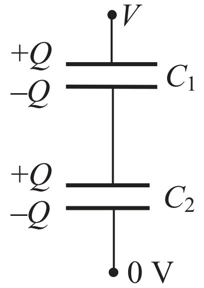
So
$$C = C_1 + C_2.$$

For N capacitors in parallel,
$$\boxed{C = C_1 + C_2 + \cdots + C_N}. \qquad [3]$$

(In a parallel combination, the total capacitance is always greater than any individual capacitance.)

Capacitors in Series

Two components are connected in series when they share a common terminal, as shown. When connected to an external potential, a charge Q will flow onto the capacitors as shown. Note that since there is no *external* connection to the centre point of the circuit, the *total* charge on the lower plate of C_1 and the upper plate of C_2 must be zero. Therefore both capacitors must carry the same charge, Q. The voltage across the circuit is just the sum of the voltages across the individual capacitors, $V = V_1 + V_2$.



But from [1], $V_1 = \frac{Q_1}{C_1} = \frac{Q}{C_1}$, etc

So
$$V = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C}.$$

and therefore
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or for N capacitors in series,
$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}}. \qquad [4]$$

(In a series combination, the total capacitance is always less than the smallest individual capacitance.)

<u>Energy Stored in a Capacitor</u>

We have seen that when charge is at an electrostatic potential, it possesses potential energy – that is, the ability to do work. A charged capacitor must therefore store energy, as well as charge. The work required to add an infinitesimal charge dq to a capacitor already carrying charge q is $\mathrm{d}W = V\,\mathrm{d}q = \frac{q}{C}\mathrm{d}q$. The total work done to charge a capacitor to charge Q is therefore

$$W = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{Q^2}{2C}.$$

This is the electrical potential energy U stored in the capacitor. Using equation [1], this can be expressed in a variety of ways.

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}. \qquad [5]$$

<u>The Energy Density of an Electric Field</u>

We can consider the energy stored in a capacitor (or other system of charges) either as being due to charges held at potentials or as energy stored in an extended electric field.

Writing $U = \frac{1}{2}CV^2$ and $V = Ed$, and using [2] $C = \frac{\varepsilon_0 A}{d}$, we have

$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 AdE^2.$$

But Ad is the volume between the plates, that is the volume where the electric field exists. The energy density, that is the energy per unit volume, is therefore just

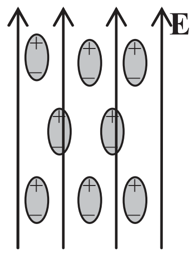
$$\boxed{u = \tfrac{1}{2}\varepsilon_0 E^2}. \qquad [6]$$

This expression is not only valid for the parallel plate capacitor, or indeed for any other sort of capacitor. It is the general expression for the energy density in an electric field.

Dielectrics

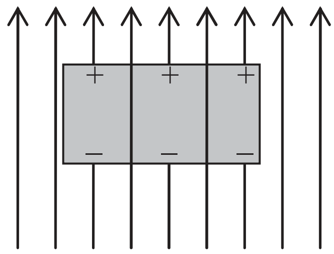
The capacitance of a capacitor could obviously be increased if we could reduce the potential at which a given charge is stored. This could be achieved if the electric field was reduced. Is there some way this can be done, for a given geometry of capacitor? The answer is that it can, if a material known as a dielectric is introduced into the gap between the capacitor plates. We have already seen that a conductor contains free charges, and that in an external electric field these move until the net internal field reaches zero. (We could not fill the gap between the capacitor plates with conductor, as this would allow charge to escape from one plate to the other, and no charge would be stored!)

A dielectric is an insulator which, though neutral overall, contains *bound* charges which cannot move throughout the bulk of the material but which can be displaced slightly under the influence of an electric field. Examples might be simple atoms, in which electrons move a small distance one way while the positive nucleus is displaced slightly the other when a field is applied, or more complicated organic molecules with polar groups which are positive and negative while being bound to the molecule.



In each case, the charges move under the influence of the applied field without leaving their parent atoms or molecules, until an internal restoring force balances the force due to the external field. They therefore become microscopic dipoles, generating a dipole moment in the direction of the applied field. In most materials, the displacement, and so the size of the dipole moment, is proportional to the applied electric field.

The net effect of this is that *within the dielectric*, the material is neutral but the overall field is reduced, as the dipoles produce a field which opposes the applied field. *On the surface of the dielectric*, excess bound charges exist, and the material is said to be *polarised*. The diagram shows a slab of dielectric, polarised by an external field and so reducing the electric field within its bulk.



The electric field inside the material is reduced by a factor k known as the *dielectric constant* of the material. This means that the potential difference between the two plates of a capacitor filled with dielectric material is also reduced by k. From equation [1], $C = \frac{Q}{V}$, we therefore see that the capacitance is increased by the same factor. For a parallel plate capacitor containing dielectric material, the capacitance is therefore

$$\boxed{C = \frac{k\varepsilon_0 A}{d}}. \qquad [7]$$

(The dimensionless constant k is also sometimes known as ε_r, the relative permittivity of the material.)

Putting What You Have Learnt Into Practice

Question 5.1

A parallel plate capacitor has plates with dimensions 3 cm by 4 cm, separated by 2 mm. The plates are connected across a 60 V battery. Find (a) the capacitance; (b) the magnitude of charge on each plate; (c) the energy stored in the capacitor.

Solution

(a) The area of the plates is 12×10^{-4} m^2. The capacitance is

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.85\times10^{-12}\times12\times10^{-4}}{2\times10^{-3}} = 5.3\,\text{pF}.$$

(b) The charge stored is $\quad Q = CV = 5.31\times10^{-12}\times60 = 3.2\times10^{-10}\,\text{C}.$

(c) The stored energy is $\quad U = \frac{1}{2}CV^2 = 0.5\times5.31\times10^{-12}\times60^2 = 9.6\times10^{-9}\,\text{J}$

Question 5.2

What is the capacitance of an isolated metallic sphere of radius R = 20 cm?

Solution

The potential of the sphere when carrying a charge Q is

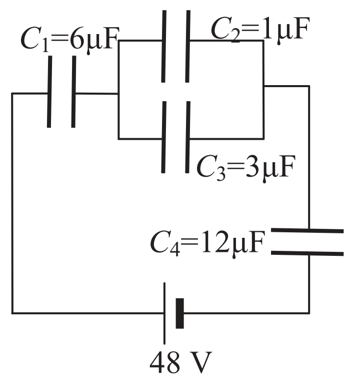
$$V = \frac{Q}{4\pi\varepsilon_0 R}.$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R = 4\pi\times8.85\times10^{-12}\times0.2 = 2.2\times10^{-11}\,\text{F}$$

Capacitance is 22 pF.

Question 5.3

For the circuit shown opposite, find: (a) the effective total capacitance; (b) the charge and potential difference for each individual capacitor.



Solution

(a) We start with the parallel sub-circuit of C_2 and C_3, and have

$$C_{23} = C_2 + C_3 = 4\,\mu\text{F}.$$

This combination is in series with the other two capacitors, so the total equivalent capacitance is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_{23}} + \frac{1}{C_4} = \frac{1}{6} + \frac{1}{4} + \frac{1}{12} = \frac{1}{2}.$$

$$C = 2\ \mu\text{F}.$$

(b) Capacitors in series have the same charge, and this is also the charge Q on the overall equivalent capacitance.

$$Q = CV = 2\times10^{-6}\times48 = 96\,\mu\text{C}$$

So $\quad Q_1 = Q_4 = Q_{23} = 96\,\mu\text{C}$

Since C_2 and C_3 have a common potential difference across their terminals, the charge they carry will be proportional to their capacitance, so ¼ will be on C_2, and ¾ on C_3.

$$Q_2 = 24\,\mu\text{C}; \qquad Q_3 = 72\,\mu\text{C}$$

We find the voltages from

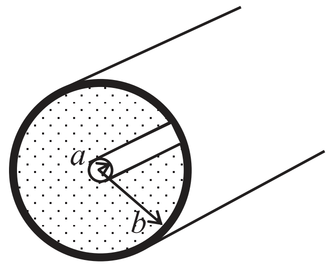
$$V_1 = \frac{Q_1}{C_1} = \frac{96\times10^{-6}}{6\times10^{-6}} = 16\,\text{V}; \qquad V_4 = \frac{Q_4}{C_4} = \frac{96\times10^{-6}}{12\times10^{-6}} = 8\,\text{V}$$

$$V_2 = \frac{Q_2}{C_2} = \frac{24\times10^{-6}}{1\times10^{-6}} = 24\,\text{V} = V_3 = \frac{Q_3}{C_3} = \frac{72\times10^{-6}}{3\times10^{-6}} = V_{23}$$

(As a check, $V_1 + V_{23} + V_4 = 16 + 24 + 8 = 48\,\text{V}$ as required.)

Question 5.4 – Cylindrical Capacitor

What is the capacitance of a long cylindrical (coaxial) cable of inner radius a, outer radius b and length L as shown? How is the capacitance changed if the insulation between the conductors is plastic with a dielectric constant k?



Solution

Consider a charge $+Q$ on the inner conductor, and $-Q$ on the outer. Choose as a Gaussian surfaces a cylinder of length L and radius r, between the other cylinders. From Gauss's law we have

$$\oint \mathbf{E}\cdot \text{dA} = \frac{Q}{\varepsilon_0}.$$

$$E\times 2\pi r\times L = \frac{Q}{\varepsilon_0} \qquad \Rightarrow \qquad E = \frac{Q}{2\pi\varepsilon_0 rL}.$$

To find the potential difference between the conductors, we integrate over a radial path from the outer to the inner conductor

$$V = -\int_b^a E\,\text{d}r = -\int_b^a \frac{Q\,\text{d}r}{2\pi\varepsilon_0 rL} = -\frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{a}{b}\right) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

Since $C = \dfrac{Q}{V}$, we have $$C = \frac{2\pi\varepsilon_0 L}{\ln\left(b/a\right)}.$$

If an insulator of dielectric constant k fills the space between the conductors, the capacitance is just increased by a factor k.

Question 5.5 – Spherical Capacitor

What is the capacitance of two concentric spherical conducting shells of inner radius a and outer radius b?

Solution

We take the same approach as in question 5.4, considering a charge $+Q$ on the inner conductor, and $-Q$ on the outer sphere. For a Gaussian surface, we consider a concentric sphere of radius r, where $a < r < b$. Applying Gauss's law yields

$$E\times 4\pi r^2 = \frac{Q}{\varepsilon_0} \qquad \Rightarrow \qquad E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Integrating from outer to inner conductor to find the potential difference we get

$$V = -\int_b^a E\,\mathrm{d}r = -\int_b^a \frac{Q\,\mathrm{d}r}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{b-a}{ab}.$$

Then from $C = \dfrac{Q}{V}$, we have $$C = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$

Problems from Young & Freedman for Topic 5:
Try to do exercises 24.1 to 24.44 and 21.47 to 24.75. The later problems are more challenging. (Numerical answers to odd-numbered questions are available at the back of the book.)