PHY102 Electricity
Topic 3 (Lectures 4 & 5) – Gauss's Law

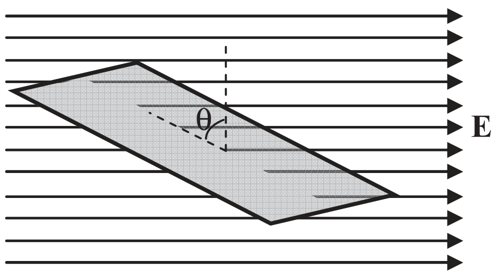
In this topic, we will cover:
1) Electric Flux
2) Gauss's Law, relating flux to enclosed charge
3) Electric Fields and Conductors revisited

Reading from Young & Freedman:
For this topic, read chapter 22.

Introduction

In the last topic, we saw various important properties of electric fields. By the principle of superposition, we can calculate the electric field due to a distributed charge. However, the integral needed to evaluate this may be tricky, and in cases where we can recognise the symmetry of the field, there may be a much easier method. We saw that the field strength is equal to the density of field lines, and we also saw that the number of lines starting on a charge is proportional to the magnitude of that charge. If we can define the constant of proportionality, we can use these two facts to determine the electric field at different points in space. The mathematician Carl Gauss related the flux, or flow of field lines through a closed surface, with the total charge enclosed within it. In fact, Gauss's Law is not restricted to electrostatics; it can also be used to relate gravitational field to mass.

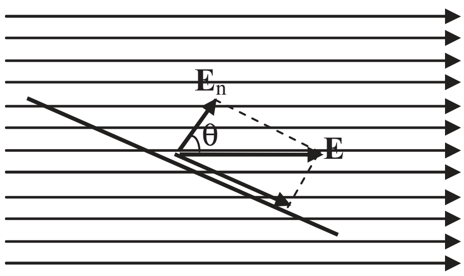
Electric Flux



Consider a rectangular surface of area A immersed in a uniform electric field E. If the surface is perpendicular to the field, the total flux cut by the surface, Φ, is just EA. If the surface is inclined as shown, we have two ways of looking at the situation. The *projected area* perpendicular to the field is now $A\cos\theta$, so $\Phi = EA\cos\theta$. We arrive at the same result if we resolve the electric field into components parallel to and normal (perpendicular) to the surface. The normal component E_n is equal to $E\cos\theta$, so the flux through the surface is just

$$\Phi = E_n A = EA\cos\theta . \quad [1]$$

Note that θ is the angle between the lines of electric field and the normal to the surface.



We therefore have two equivalent definitions of electric flux:

The electric flux through an area is equal to the number of field lines intercepted by that area.

The electric flux through an area is the product of the area with the normal component of the field through that area.

We can usefully express [1] in vector notation. If **A** is a vector representing the surface whose magnitude is equal to the area and direction is given by the normal to the surface, then Φ is given by the dot-product

$$\Phi = \mathbf{E}\cdot\mathbf{A} . \qquad [2]$$

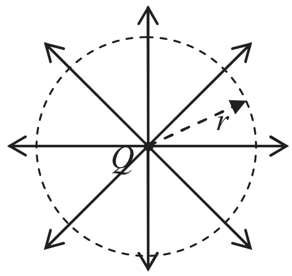
In general, the electric field will not be uniform, and the surface we need to consider need not be plane. However we can divide the surface into infinitesimal elements d**A** within which the field will be approximately constant. The flux through an element will be given by $d\Phi = \mathbf{E}\cdot d\mathbf{A}$ and the total flux through the surface is given by integrating over the surface:

$$\boxed{\Phi = \int \mathbf{E}\cdot d\mathbf{A} = \int E\cos\theta\, dA}. \qquad [3]$$

Field lines passing one direction through a surface give a positive contribution to the flux, while those going in the opposite direction will give a negative contribution. If we consider a closed surface, such as a sphere, we define the net flux as the number of field lines emerging from the surface minus the number entering it. (If more lines enter than leave, the net flux is negative.) An important result is that since field lines always start and end on a charge, if there is no charge within a surface, the net flux through that surface must be zero.

<u>Gauss's Law</u>

Consider a point positive charge Q surrounded by a spherical so-called *Gaussian surface*, as shown in the figure. By symmetry, the electric field has the same value at all points on this imaginary sphere, and its direction is always perpendicular to the surface, so parallel to the vector elements d**A** of the surface. The total flux through the closed surface is therefore



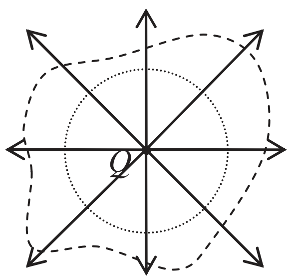
$$\Phi = \oint E\, dA = E\oint dA = E\times 4\pi r^2 . \qquad [4]$$

Here $\oint$ represents an integral over a *closed* surface. Since E is constant, it has been taken out of the integral, and the integrated area is just the surface area of a sphere.

From Coulomb's law, we have $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$ so $\Phi = \dfrac{Q}{4\pi\varepsilon_0 r^2}4\pi r^2 = \dfrac{Q}{\varepsilon_0}$.

The flux through this closed surface is $1/\varepsilon_0$ times the charge enclosed within the surface. The number of field lines emerging from a charge Q is therefore Q/ε_0. Note that the above result does not depend on the radius of the sphere; since $E \propto 1/r^2$ and $A \propto r^2$ their product remains constant.

We can also extend the argument to an arbitrary, non-spherical surface. Consider the irregular surface illustrated. We know that the flux through the enclosed dotted spherical surface is Q/ε_0. Since there is no charge present *between* the inner spherical and outer arbitrary surface, we know that the net flux through the combination of these two surfaces must be zero. Therefore the flux through the outer surface must also be equal to Q/ε_0.



Gauss's law states that:

> *If the volume within an arbitrary closed surface contains a net charge Q, then the electric flux through the surface is Q/ε_0.*

To summarise:

1) The net flux through any closed surface surrounding a point charge Q is given by Q/ε_0, independent of the shape of the surface.
2) The net flux through a closed surface that surrounds no net charge is zero.
3) The electric field due to a number of charges is the vector sum of the fields due to each charge. The flux through a closed surface is therefore given by

$$\Phi = \oint \mathbf{E}\cdot d\mathbf{A} = \oint (\mathbf{E}_1 + \mathbf{E}_2 + \cdots)\cdot d\mathbf{A} .$$

Using Gauss's Law

Gauss's law is particularly helpful in determining the electrostatic field when the charge distribution has a high degree of symmetry. We will see examples of this at the end of these notes. In choosing the appropriate Gaussian surface, the following points should always be considered:

1) Use the symmetry of the charge distribution to determine the pattern of field lines.

2) Choose a Gaussian surface so that **E** is either parallel to d**A** or perpendicular to it, if possible.

3) Where **E** is parallel to d**A**, the magnitude of *E* must be constant over this part of the surface.

Electric Conductors in Electrostatic Equilibrium

We saw in the last topic that the electrons in a conductor are free to move within the body of the conductor in response to an applied field. They will move towards the surfaces of the conductor, accumulating a net negative charge at one side and leaving a net positive charge at the other, until the resulting electric field within the conductor (due to the combination of the applied field and the displaced charge) is zero. (This condition is known as *electrostatic equilibrium*.) We can now show that the net electric charge can reside *only* on the surface of a conductor. Consider an arbitrary Gaussian surface drawn entirely within the body of the conductor. Within the conductor there is no field, so the net flux through the surface must be zero. Therefore, by Gauss's law, the net charge within the Gaussian surface must be zero. Net charge cannot reside within the conductor. (Charge *can* reside on the surface of the conductor, because here a surrounding Gaussian surface would be partly within the conductor and partly outside. The electric field *outside* the conductor will *not* be zero.) In summary:

1) The electric field is zero everywhere inside a conductor.

2) When an isolated conductor carries a charge, it resides on the surface of the conductor.

3) The electric field just outside a charged conductor is always perpendicular to its surface, and has a magnitude σ/ε_0 where σ is the surface charge density at that point.

4) On an irregularly shaped conductor, the surface charge density is greatest in regions where the radius of curvature of the surface is smallest.

Putting What You Have Learnt Into Practice

Question 3.1
What is the electric flux through a sphere of radius 1.0 m containing a charge of +1 μC at its centre?

Solution by Coulomb's Law
The electric field at a distance of 1 m is radially outwards and, as determined by Coulomb's law, its magnitude is

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{1\times10^{-6}}{4\pi\times8.85\times10^{-12}\times1.0^2} = 8.99\times10^3\ \mathrm{N\,C^{-1}}$$

The area of the sphere is $A = 4\pi r^2 = 4\pi\,1.0^2 = 12.6\,\mathrm{m}^2$.

So since the field and surface are everywhere perpendicular, the flux is

$$\boxed{\Phi = EA = 8.99\times10^3\times12.6 = 1.13\times10^5\ \mathrm{N\,m^2\,C^{-1}}.}$$

Solution by Gauss's Law
For *any* surface, the flux is

$$\boxed{\Phi = \frac{Q}{\varepsilon_0} = \frac{1\times10^{-6}}{8.85\times10^{-12}} = 1.13\times10^5\ \mathrm{N\,m^2\,C^{-1}}.}$$

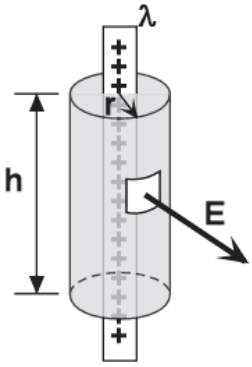
Question 3.2
Find the electric field due to a point charge q using Gauss's law

Solution
For a point charge, the electric field must be spherically symmetric. Consider therefore a Gaussian surface which is a sphere centred on the charge. The electric field at the surface of the sphere must therefore be constant in magnitude and radial. By Gauss's law

$$\Phi = \oint \mathbf{E}\cdot \mathrm{d}\mathbf{A} = \int E\mathrm{d}A = E\times4\pi r^2 = q/\varepsilon_0$$

$$\boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}.}$$

Question 3.3
What is the electric field due to an infinitely long wire carrying a linear charge density λ C m^{-1}?



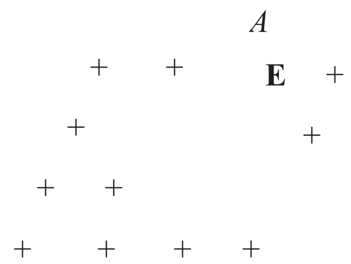
Solution
In this case, the electric field must be spread radially in two dimensions. A suitable Gaussian surface is therefore a cylinder, as shown. Once again, the electric field will be perpendicular to the curved surface of the cylinder. By Gauss's law, the electric field multiplied by the surface area must equal the charge enclosed divided by ε_0. (Since the ends of the cylinder are parallel to the field, they provide no contribution to the flux.) If we consider a length h of the wire and cylinder, we therefore have

$$\Phi = EA = E \times 2\pi r h = \frac{q}{\varepsilon_0} = \frac{\lambda h}{\varepsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}.}$$

Contrast the $1/r$ dependence of E here with $1/r^2$ for the point charge.

Question 3.4
What is the electric field due to a large uniform sheet of charge with a surface charge density of σ C m^{-2}?



Solution
Once again, we can use symmetry to see that the electric field must be uniform and everywhere perpendicular to the sheet of charge. For a Gaussian surface, we can take a cylinder with bases of area A on either side of the sheet, as shown. In this case, there are no lines of flux cutting the curved surface of the cylinder, and we only need to consider the field passing through the two ends. These leads to:

$$\Phi = 2EA = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}.$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}.}$$

(This is the result already obtained in Question 2.5 of Topic 2.)

Question 3.5
A charge Q is distributed uniformly throughout a sphere of radius R. Calculate the electric field (a) at a point outside the sphere; (b) at a point within the sphere.

Solution
As in questions 3.1 & 3.2, we have spherical symmetry and so can consider a spherical Gaussian surface, of radius r, centred on the centre of the charge distribution. Since the electric field will everywhere be perpendicular to the surface, Gauss's law tells us that

$$4\pi r^2 E = \frac{Q_{\text{within}}}{\varepsilon_0}$$

where Q_{within} is the charge contained inside the surface.

(a) If $r > R$, then $Q_{\text{within}} = Q$ as all the charge Q is contained inside the Gaussian surface. Hence

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

identical to the field when Q is concentrated at a point at the centre of the sphere.

(b) The charge density inside the sphere is $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$. For $r < R$, the charge contained inside the Gaussian surface is therefore $Q_{\text{within}} = \rho \times \frac{4}{3}\pi r^3 = Q\frac{r^3}{R^3}$.

Hence

$$\boxed{E = \frac{Q\frac{r^3}{R^3}}{4\pi\varepsilon_0 r^2} = \frac{Qr}{4\pi\varepsilon_0 R^3}.}$$

In other words, the field rises linearly from 0 at the centre to $E = \frac{Q}{4\pi\varepsilon_0 R^2}$ at the surface. This is identical to the value calculated in part (a) immediately *outside* the surface.

Problems from Young & Freedman for Topic 3:
Try to do exercises 22.1 to 21.65. The later problems are more challenging. (Numerical answers to odd-numbered questions are available at the back of the book.)