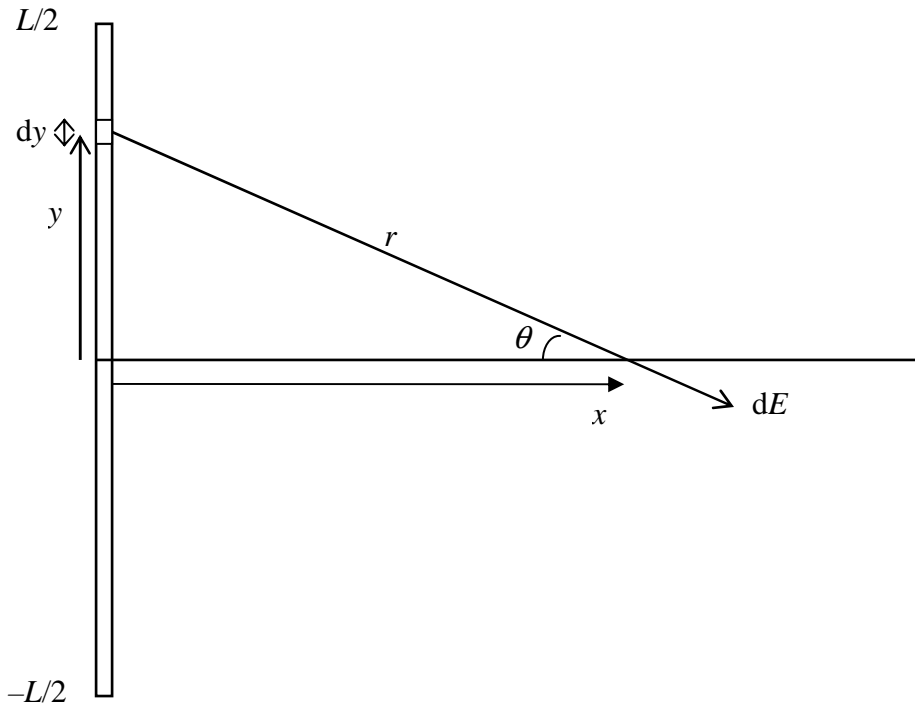


PHY102 Electricity

Example Calculation (from lecture)

A rod of length L carries a charge Q distributed uniformly along its length. If it is centred on the origin and oriented along the y -axis, what is the resulting electric field at points on the x -axis?



The linear charge density along the rod is $\lambda = Q/L$.

To calculate the overall field at a point on the x -axis, we divide the rod into infinitesimal slices dy , find the contribution to E from each of these, and sum (integrate) using the principle of superposition.

Charge in slice dy is λdy .

$$dE \text{ due to } dy = \frac{\lambda dy}{4\pi\epsilon_0 r^2}.$$

By symmetry, the overall E field on the x -axis must be parallel to x . (Contributions in the negative y direction due to slices above the x -axis as shown will be cancelled by those in the positive y direction due to slices below the x -axis.)

$$dE_x = \frac{\lambda dy \cos \theta}{4\pi\epsilon_0 r^2} \quad \Rightarrow \quad E = \int_{-L/2}^{L/2} \frac{\lambda dy \cos \theta}{4\pi\epsilon_0 r^2} \quad (1)$$

We have now done all the physics, but have some work to do on the maths! The integral contains three variables, y , r and θ , which are all interdependent. There is also no explicit dependence on x , though we want to find the field a distance x from the origin. We must therefore change variables.

$$x = r \cos \theta \quad \Rightarrow \quad r = x / \cos \theta$$

$$y = x \tan \theta \quad \Rightarrow \quad \frac{dy}{d\theta} = x \sec^2 \theta = \frac{x}{\cos^2 \theta} \quad (*) \quad \text{or} \quad dy = \frac{x}{\cos^2 \theta} d\theta$$

Substitute these into (1):

$$\Rightarrow E = \frac{\lambda}{4\pi\epsilon_0} \int \frac{\cos^2 \theta}{x^2} \cos \theta \frac{xd\theta}{\cos^2 \theta} = \frac{\lambda}{4\pi\epsilon_0 x} \int \cos \theta d\theta = \frac{\lambda}{4\pi\epsilon_0 x} [\sin \theta].$$

We now need to put in the correct limits. When $y = L/2$, $\sin \theta = \frac{L/2}{\sqrt{x^2 + (L/2)^2}}$, and when

$$y = -L/2, \sin \theta = \frac{-L/2}{\sqrt{x^2 + (L/2)^2}}.$$

$$\text{So } E = \frac{\lambda}{4\pi\epsilon_0 x} \left(\frac{L/2}{\sqrt{x^2 + (L/2)^2}} - \frac{-L/2}{\sqrt{x^2 + (L/2)^2}} \right) = \frac{Q}{4\pi\epsilon_0 x \sqrt{x^2 + (L/2)^2}}$$

The total electric field is $E = \frac{Q}{4\pi\epsilon_0 x \sqrt{x^2 + (L/2)^2}}$ in the x direction.

It is always useful to apply a “sanity check” (where this does not involve too much work!). The electric field at a large distance from the rod should not depend on the exact arrangement of the charge. If we consider the limit of the above expression when $x \gg L$, we obtain

$$E \rightarrow \frac{Q}{4\pi\epsilon_0 x^2}, \text{ exactly as expected!}$$

(*) If you have forgotten that $\frac{d \tan \theta}{d\theta} = \sec^2 \theta$, you can easily derive this using the identity

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \text{ together with the quotient rule for differentiation.}$$

For further calculations of electric field due to extended charge distributions, see the application section of the handout.