Background Mathematics for PHY102 Electricity

Introduction

An important aspect of your first year Physics course is learning to apply mathematical techniques to the physical laws governing a situation. You will cover all the required mathematics in more detail in maths courses during the year, but there are some topics which you may meet in the physics lectures before you have covered the material in the mathematics lectures. A summary of some important areas is provided below.

1 Vectors

Vectors are used to represent physical quantities which have both a size (or magnitude) and a direction. Examples include force, displacement, electric field, etc.

Notation

It is important to distinguish between scalar and vector quantities. The symbol used to represent electric field, for example, is normally "*E*". We would use E to denote the magnitude (or size) of the field, and $\mathbf{E}$, $\underline{E}$ or $\vec{E}$ can be used for the vector, which includes the size and the direction. In my printed notes, I use bold letters ($\mathbf{E}$) for vectors, while in hand-written material (e.g. on the board) I will underline the symbol ($\underline{E}$). The third notation, with an arrow, is used in many text books. When we wish to emphasise that we are only taking the magnitude of a vector, we can use modulus signs: $E = |\mathbf{E}|$.

Unit Vectors

Unit vectors, as their name implies, have a length of one unit, and are used to indicate direction only. If the displacement of point A from point O is $\mathbf{r}$ (or $\underline{r}$), then $\hat{\mathbf{r}}$ or $\underline{\hat{r}}$ gives the direction of A measured from O. Thus $\mathbf{r} = r\,\hat{\mathbf{r}}$ (the vector is equal to its magnitude multiplied by its direction).

Components

It is sometimes useful to express a vector in terms of its components, often those parallel to the Cartesian coordinates of x, y, z. So we could say $\mathbf{r} = (3.1, 2.9, 0.0)\,\mathrm{m}$. (Note the need to specify the units!) An alternative way to express the same thing is to use the special unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ which give the directions of the x, y and z axes respectively. (Exceptionally, these unit vectors are not normally written with a "hat"!) So we can also write $\mathbf{r} = (3.1\mathbf{i} + 2.9\mathbf{j} + 0.0\mathbf{k})\,\mathrm{m}$.

We can also find the component in some arbitrary direction. If the angle between an electric field, $\mathbf{E}$, and the direction of interest is θ, then the component of $\mathbf{E}$ in that direction is $E\cos\theta$.

Products of Vectors

There are several ways of multiplying vectors. If we simply multiply a vector, $\mathbf{V}$, by a scalar, S, each component is scaled up by the same value S, and the direction is unchanged. The result is $S\,\mathbf{V}$.

The simplest way of multiplying two vectors together yields the scalar product or dot product. As the name implies, this has just a value, not a direction. If the angle between vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is θ, then the scalar product is given by $\mathbf{V}_1 \cdot \mathbf{V}_2 = V_1 V_2 \cos\theta$. (For example, work done W can be calculated from force $\mathbf{F}$ and displacement $\mathbf{d}$ by $W = \mathbf{F}\cdot\mathbf{d} = F\,d\cos\theta$ – force multiplied by the displacement *in the direction of the force*.)

The vector product or cross product between two vectors yields a vector result which is perpendicular to the plane containing both the original vectors. (As might be expected, if the vectors are parallel, and so do not define a unique plane, their vector product is zero.) Considering the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ at angle θ again, the vector product is written as $\mathbf{V}_1 \times \mathbf{V}_2$ and has a magnitude $V_1 V_2 \sin\theta$.

2 Binomial Theorem and Binomial Expansion

You should be familiar with the binomial expansion:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots .$$

This is easily verified for the case when n is a positive integer.

Most physical applications relate to cases where $b << a$, and n may be fractional or negative. In this case, we can write

$$(a+b)^n = a^n\left(1+\left(\frac{b}{a}\right)^n\right) = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}\right)^3+\cdots\right).$$

If $b << a$, then it follows that $\frac{b}{a} << 1$ and $\left(\frac{b}{a}\right)^2 << \frac{b}{a}$ etc, so terms to the right rapidly become negligible. The exact number of terms which must be retained depends on the problem being considered, but the binomial expansion allows a very good approximation to the exact result to be obtained.

3 Volume Integrals

There are a number of cases where you will need to integrate not over 1 dimension but over volume. For example, for an object of constant density ρ, the mass m is just density times volume V, $m = \rho V$. However, if the density varies from region to region, the mass must be found by integrating the density over the volume, $m = \int \rho\, \mathrm{d}V$. In a general case, this (3-dimensional) integral may be hard to do, but if the quantity to be integrated has an appropriate symmetry, the calculation can be performed easily.

Spherical Symmetry

If in the above example the density has spherical symmetry, it only depends on the (scalar) distance r from the centre. $\rho(r)$ has a constant value over a surface with constant r, i.e. over the surface of a sphere. The volume can therefore be divided up into a set of nested spherical shells, each of area $4\pi r^2$ and thickness dr, so of volume $\mathrm{d}V = 4\pi r^2\, \mathrm{d}r$. Our integral therefore becomes $m = \int \rho(r)\, \mathrm{d}V = \int \rho(r)\, 4\pi r^2 \mathrm{d}r$ between the appropriate limits.

Cylindrical Symmetry

If the quantity such as density has cylindrical symmetry, it only depends on the distance r from the axis of the cylinder (e.g. x or z coordinate axis). $\rho(r)$ has a constant value over a surface with constant r, i.e. over the curved surface of a cylinder. If the cylinder has length ℓ, the volume can be divided up into a set of nested cylindrical shells, each of area $2\pi r\ell$ and thickness dr, so of volume $\mathrm{d}V = 2\pi r\ell\, \mathrm{d}r$. The integral therefore becomes $m = \int \rho(r)\, \mathrm{d}V = \int \rho(r)\, 2\pi r\ell \mathrm{d}r$ between the appropriate limits.

(In the E & M course, it will more often be charge and charge-density, rather than mass and matter-density, that we will be considering. The principle, however, remains the same.)